Completeness Theorems and λ-calculus

Thierry Coquand

Apr. 23, 2005

Content of the talk

We explain how to discover some variants of Hindley's completeness theorem (1983) via analysing proof theory of impredicative systems

We present some remarks about the role of completeness theorems in proof theory and constructive mathematics

For λ-calculus: completeness w.r.t. Kripke models versus ordinary set-theoretical models

Analysis of impredicativity

The problem of impredicativity was formulated through discussions between Poincaré and Russell

Poincaré argued that impredicative definitions are “circular” and should be avoided

More than 100 years later, this is still one of the most important problem in proof theory

Analysis of impredicativity

A typical impredicative definition is Leibniz' definition of equality

"a is equal to b" is defined as

$\forall X.X(a) \Rightarrow X(b)$

Here X ranges over *all* predicates and thus can be the predicate

$X(u) =_{def} \qquad u$ is equal to a

This is circular

Analysis of impredicativity

If we have already an equality relation $=$ like in first-order equational logic, one can prove that

$$a = b \;\Leftrightarrow\; (\forall X.X(a) \Rightarrow X(b))$$

An higher-order impredicative defined relation can be equivalent to a first-order definition relation

Russell's *axiom of reducibility* (Principia Mathematica) says precisely that this should be the case for any impredicative definition. But this amounts precisely to accept any impredicative definitions.

Problem with impredicativity

Russell's example

"*a typical Englishman is one who possesses all the properties possessed by a majority of Englishmen*"

There is a potential problem: when one speaks of "all properties" one should not really mean "all properties" but only "all properties that do not refer to a totality of all properties"

Cf. definition of random numbers

Proof theory and impredicativity

To analyse impredicativity (in particular, does it lead to a paradox or not?) has been one of the main goal of proof theory

No definite conclusion yet, but many partial results that confirm the extreme proof-theoretic strength of impredicative definitions

Proof theory and impredicativity

The basic *completeness theorem* of first-order logic states that a closed first-order formula is true iff it is provable

As emphasized by Girard, by closed, we mean that it has no free variable and we should count the predicate, relation symbols as variables

The "correct" statement should involve a second-order quantification Π^1

$\forall X.\forall a.X(a) \Rightarrow X(a)$

Proof theory and impredicativity

Completeness theorems "explain" some cases of impredicative quantification: the meaning of

$$\forall X.\forall a.X(a) \Rightarrow X(a)$$

to be *true* is that it is *provable*, and a proof is a finite concrete object

Proof theory and impredicativity

Usual completeness theorem concerns first-order logic statements

Gentzen introduced sequent calculus and his first version of his consistency proof for arithmetic introduces (implicitely) ω-logic and a possible constructive explanation of classical validity for arithmetical statements

ω-logic was introduced explicitely by Novikov (1943) and independently by Schütte. It provides a constructive explanation of statements $\phi(X)$ with a predicate or relation variable X with quantifiers ranging over natural numbers

Proof theory and impredicativity

For example, the formula $(\wedge_n X(n)) \vee \vee_n \neg X(n)$ because for each n_0 the formula

$$X(n_0) \vee \vee_n \neg X(n)$$

is valid

The fact that such a statement is *true* iff it is *provable* is known as *Henkin-Orey* completeness theorem for ω-logic

In this logic, a proof tree is a well-founded countably branching tree

Proof theory and impredicativity

A suggestive way to interpret these results is the following

We can explain the meaning of a statement $\forall X.\phi(X)$ without relying on quantification over all subsets of $\mathbb{N}$ by explaining how to prove $\phi(X)$

In this way we have an explanation of some impredicative quantifications

Π^1 definitions

This interpretation of Π^1 definitions is essential in recent work on constructive representation of infinite objects

By replacing universal quantification over predicates by provability (with a free variable predicate) one can represent faithfully classical infinite objects in constructive mathematics

This explanation which replaces a *semantical* notion (quantification over all subsets) by a *syntactical* (free variable and provability) is in the spirit of Hilbert's program

Proof theory and impredicativity

This reduction of impredicative definitions to inductive definitions was first obtained by G. Takeuti in the 1950s

Formulated a sequent calculus for second-order logic

$$\frac{\Gamma, \forall X.\phi(X), \phi(T) \vdash \Delta}{\Gamma, \forall X.\phi(X) \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi(X), \Delta}{\Gamma \vdash \forall X.\phi(X), \Delta}$$

Proof theory and impredicativity

Takeuti conjectured that cut-elimination holds for this calculus

Takeuti proved also that *cut-elimination* for his sequent formulation of second-order *logic* implies *consistency* of second-order *arithmetic*

He could proved cut-elimination for *restricted subsystem*

The basic one is where we can only build $\forall X.\phi(X)$ if $\phi(X)$ is first-order

Proof theory and impredicativity

To prove that cut-elimination implies consistency of second-order *arithmetic*

Notice that a cut-free proof of a first-order statement has to be first-order

Take the first-order theory

$$S(x) = S(y) \Rightarrow x = y, \qquad S(x) \neq 0$$

This theory is consistent, and implies the infinity axiom

We can then interpret second-order arithmetic in this theory

Proof theory and impredicativity

His method was indirect: first express impredicative definitions in a sequent calculus, and then prove cut-elimination using ordinals

Justify then the use of ordinals using only inductive definitions in an intuitionistic way

This is one of the main result of proof theory (1950s): reduction of Π^1_1-impredicativity to inductive definitions (well-founded trees)

The Ω-rule

This was later simplified by Buchholz

The basic idea is that completeness theorems gives an explanation of statements where Π^1 statements appear *positively*

$\phi \Rightarrow \forall X.\psi(X)$ will hold iff $\phi \Rightarrow \psi(X)$ is provable with X variable

The Ω-rule

The problem is to interpret statements with *negative* occurences of Π^1 statements

$(\forall X.\psi(X)) \Rightarrow \phi$

It does not work to explain it as

"if $\psi(X)$ is provable then ϕ is provable"

"if $\vdash \psi(X)$ then $\vdash \phi$"

The Ω-rule

One needs to explain transitivity of implication, and this does not work if

$(\forall X.\psi(X)) \Rightarrow \phi$ is $\vdash \psi(X)$ implies $\vdash \phi$

$\delta \Rightarrow (\forall X.\psi(X))$ is $\vdash \delta \Rightarrow \psi(X)$

Indeed, from

"$\vdash \psi(X)$ implies $\vdash \phi$" and

$\vdash \delta \Rightarrow \psi(X)$

we cannot conclude $\vdash \delta \Rightarrow \phi$

The Ω-rule

Instead Buchholz introduced the Ω-rule:

$(\forall X.\psi(X)) \Rightarrow \phi$

is explained by, *for all* δ

$\vdash \delta \Rightarrow \psi(X)$ implies $\vdash \delta \Rightarrow \phi$

where δ ranges over formulae without second-order quantification

In this way $\delta \Rightarrow \forall X.\psi(X)$ and $(\forall X.\psi(X)) \Rightarrow \phi$ imply $\delta \Rightarrow \phi$

Proof theory and impredicativity

We obtain in this way a complete explanation of some limited form of impredicativity, by giving a model of the fragment of second-order logic

In this fragment, the impredicative quantifications are all of the form $\forall X.\phi(X)$ where $\phi(X)$ is first-order

One uses the Ω-rule; implicitely, this amounts to working in a Kripke model

Proof theory and impredicativity

There is a crucial difference between the explanation of

$\forall X.\psi(X)$ valid iff $\vdash \psi(X)$

which is *complete* for Π^1 statement, and the explanation of formulae with negative occurence of Π^1 formula such as

$(\forall X.\psi(X)) \Rightarrow \phi$

which cannot be complete

As stressed by Girard, this is a formulation of Gödel's incompleteness theorem

completeness fails outside Π^1

Proof theory and impredicativity

Usually incompleteness is stated for Π^0_1 statements

$\forall n.\phi(n)$

If we formulate it in second-order *logic* we get a statement of the form

$\forall x.N(x) \Rightarrow \phi(x)$

where $N(u)$ is a Π^1 formula

$N(u) = \forall X.X(0) \Rightarrow (\forall y.X(y) \Rightarrow X(S(y))) \Rightarrow X(u)$

System F

An elegant formalism in which impredicativity is represented in a pure way is Girard's system F (1970)

$$U \ ::= \ X \mid U \rightarrow U \mid \Pi X.U$$

$$M \ ::= \ x \mid M \ M \mid \lambda x.M \mid \lambda X.M \mid M \ U$$

$\lambda X.\lambda x.\lambda x$ is of type $\Pi X.X \rightarrow X$ and can be applied to its own type

Circularity in system F

What is the reducibility predicate for $\Pi X.X \to X$?

Should be: M is reducible iff for all U the term $M\ U$ is reducible at type $U \to U$

This is circular: U may be $\Pi X.X \to X$ itself

Circularity in system F

Girard broke the circularity by introducing the notion of reducibility candidate

M is reducible at type $\Pi X.X \rightarrow X$ iff for all U *and all* reducibility candidate C at type U the term $M\ U$ is reducible at type $U \rightarrow U$, i.e. $C(M\ U\ N)$ if $C(N)$

One can argue that this only rejects the circularity at the meta-level but that the circularity is still there

Circularity in system F

A simpler formulation is obtained by taking untyped lambda terms

$U \; ::= \; X \mid U \to U \mid \Pi X.U$

$M \; ::= \; x \mid M \; M \mid \lambda x.M$

We interpret *types* as *sets* $A, B \cdots \subseteq \Lambda$ of λ-terms

Circularity in system F

$A \to B = \{M \in \Lambda \mid N \in A \Rightarrow M\ N \in B\}$

$\Pi X.T(X)$ is interpreted by intersection

$[\![\Pi X.X \to X]\!] = \cap_A.(A \to A)$

We see the circularity: we build a subset of Λ by taking an intersection over *all* subsets of Λ

Circularity in system F

What happens if one restricts system F to the subsystem F_0 where one allows only to form $\Pi X.T(X)$ if $T(X)$ is built only with X and $\rightarrow$??

Can one use the techniques of proof theory and give a predicative normalisation proof for this fragment?

I. Takeuti gave such a proof in 1993, following G. Takeuti, and provided an ordinal analysis of normalisation, with ordinals $< \epsilon_0$

In TLCA 2001, T. Altenkirch and I gave an analysis following the Ω-rule, but the argument can be much simplified (following suggestions of Buchholz)

Circularity in system F

Theorem: *In the restricted system F_0 the functions of type $N \to N$ are exactly the functions provably total in Peano arithmetic*

In several cases, an a priori impredicative intersection can be replaced by a set which has a predicative description

Circularity in system F

For instance

$\cap_A\ (A \to A)$ is $\{\lambda x.x\}$

$\cap_A\ A \to (A \to A) \to A$ is $\{\lambda x.\lambda f.f^n\ x \mid n \in \mathbb{N}\}$

Has $\cap_A\ \Phi(A)$ always a predicative description? This is not clear a priori

$\cap_A\ ((A \to A) \to A) \to A$

A typing system

$M, M', \ldots$ denote *untyped* lambda-terms

$T, T', \ldots$ denote *first-order* types

$T \ ::= \ X \mid T \to T$

$$\frac{}{x_1 : T_1, \ldots, x_k : T_k \vdash x_i : T_i} \qquad \frac{\Gamma, x : T \vdash M : T'}{\Gamma \vdash \lambda x.M : T \to T'}$$

$$\frac{\Gamma \vdash M : T \to T' \qquad \Gamma \vdash N : T}{\Gamma \vdash M \ N : T'} \qquad \frac{\Gamma \vdash M : T \qquad M =_{\beta,\eta} M'}{\Gamma \vdash M' : T}$$

Circularity in system F

If we follow the technique of the Ω-rule we get the following result: $\cap_A \; \Phi(A)$ has a predicative description if we work in the Kripke model where the worlds are *first-order contexts* and the ordering is reverse inclusion

First-order contexts $\Gamma, \ldots$ of the form $x_1 : T_1, \ldots, x_k : T_k$

Let H be the poset of downward closed sets of such worlds (truth values for the Kripke model)

A type is now interpreted as a function $A : \Lambda \rightarrow \mathsf{H}$

Circularity in system F

We write $\Gamma \Vdash M \in A$ instead of $\Gamma \in A(M)$

We have now $\Gamma \Vdash M \in A \rightarrow B$ iff

$\forall \Delta \supseteq \Gamma . \forall N. \;\; \Delta \Vdash N \in A \;\;\; \Rightarrow \;\;\; \Delta \Vdash M \; N \in B$

We can define for each T with free variables $X_1, \ldots, X_k$ its semantics $[\![T]\!]_{X_1=A_1,\ldots,X_k=A_k} : \Lambda \rightarrow \mathsf{H}$

A completeness theorem

In general we write $[\![T]\!]_{X_1=A_1,\ldots,X_k=A_k}$ the interpretation of a type $T(X_1,\ldots,X_k)$

We define $\downarrow T$ by $\Gamma \Vdash M \in \downarrow T$ iff $\Gamma \vdash M : T$

Lemma 1: $[\![T]\!]_{X_1=\downarrow X_1,\ldots,X_k=\downarrow X_k} \quad = \quad \downarrow T$

Direct by induction on T

"Yoneda lemma" $\downarrow (T_1 \rightarrow T_2) = \downarrow T_1 \rightarrow \downarrow T_2$

A completeness theorem

Lemma 1: $[\![T]\!]_{X_1=\downarrow X_1,\ldots,X_k=\downarrow X_k} \quad = \quad \downarrow T$

Lemma 2: *If* $x_1 : T_1, \ldots, x_l : T_l \vdash M : T$ *and* $\Gamma \Vdash N_1 \in [\![T_1]\!], \ldots, \Gamma \Vdash N_l \in [\![T_l]\!]$ *then* $\Gamma \Vdash M(x_1 = N_1, \ldots, x_l = N_l) \in [\![T]\!]$

Using lemma 1 and lemma 2 we get

Corollary: *If* $\Gamma \vdash M : T(X)$ *where* $T(X)$ *built from* X *and* $\rightarrow$ *and* X *not free in* Γ *then* $\Gamma \Vdash M \in [\![T]\!]_{X=A}$ *for any* $A : \Lambda \rightarrow \mathsf{H}$

A completeness theorem

Assume $T(X)$ built from X and $\rightarrow$

It is natural to define $\Gamma \Vdash M \in [\![\cap_X T(X)]\!]$ by

$$\Gamma \Vdash M \in [\![T]\!]_{X=A}$$

for all $A : \Lambda \rightarrow \mathsf{H}$

Theorem: $\Gamma \Vdash M \in [\![\cap_X T(X)]\!]$ *iff* $\Gamma \vdash M : T(X)$

A completeness theorem

This gives a predicative description even of non algebraic types such as

$T = \cap_A \; ((A \to A) \to A) \to A$

In the Kripke model, we have $\Gamma \Vdash M : T$ iff

$\Gamma \vdash M : ((X \to X) \to X) \to X$

A completeness theorem

This interpretation of impredicative comprehension is done in a metalogic where we use only arithmetical comprehension

It is standard that second-order arithmetic with arithmetical comprehension is conservative over Peano arithmetic

From this, it is possible to deduce that the functions of type $N \rightarrow N$ representable in the system F_0 are exactly the functions provably total in Peano arithmetic

Hindley's completeness

It is remarkable that, in the case of λ-calculus, the use of Kripke models, which seems so crucial for the Ω-rule, is not necessary

Theorem: (Hindley, 1983) $M \in \cap_A T(A)$ *iff* $\vdash M : T(X)$

The proof however involves a *non canonical* enumeration of types and variables

In effect it builds an infinite context, instead of working with the Kripke model where worlds are finite contexts

The proof does not involve any quantification over all subsets of an infinite set

Hindley's completeness

For instance $M \in \cap_A \ (A \to A)$ iff

$\vdash M : X \to X$

and this is the case iff $M =_{\beta,\eta} \lambda x.x$

Similarly

$M \in \cap_A \ A \to (A \to A) \to A$

iff

$\vdash M : X \to (X \to X) \to X$

Hindley's completeness

But also

$M \in \cap_A \; ((A \to A) \to A) \to A$

iff

$\vdash M : ((X \to X) \to X) \to X$

This is a *predicative* description of $\cap_A \; ((A \to A) \to A) \to A$

Extension

One can consider the hierarchy F_n of subsystems of system F

In the system F_{n+1} we allow $\Pi X.U(X)$ where X is built from X, $\rightarrow$ *and* closed types of F_n

For instance

$$\Pi X.X \rightarrow (X \rightarrow X) \rightarrow ((N \rightarrow X) \rightarrow X) \rightarrow X$$

is a type of the system F_1

Extension

Theorem: (K. Aehlig, 2003) *The terms of type $N \rightarrow N$ in the system F_n represent exactly the functions provably total in the logic of inductive definitions ID_n*

The proof follows the method of the Ω-rule

Related results

One would like to obtain normalisation by evaluation via extraction of a proof of normalisation for simply typed λ-calculus

In order to formalise this proof one uses a Kripke model or a Fraenkel-Mostowski model

How does the extracted computation compares to the set-theoretical semantics?

Π^1 definitions

For instance to express that an element u is in the intersection of all prime ideals can be expressed as (in any extension of the equational theory of rings)

$\phi(X) \Rightarrow X(u)$

where $\phi(X)$ denotes the conjunction

$X(0) \ \wedge \neg X(1) \ \wedge$

$(\forall r, s. X(rs) \Leftrightarrow (X(r) \vee X(s))) \ \wedge$

$(\forall r, s. X(r) \wedge X(s) \Rightarrow X(r+s))$

One can then prove, constructively, that this holds iff u is nilpotent even though prime ideal may fail to exist constructively