A PROPOSAL ABOUT FOUNDATIONS I

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§1 INTRODUCTION

In this paper we propose an answer to the question: "If the objects of an elementary topos are to be thought of as sets, then what are the classes?" For example, can the topos itself be seen as its own class of all sets? A satisfying answer to this question is one that would allow us to extend the topos' internal logic to these classes, and in particular quantify over them: our aim is to present a form of higher-order logic that will do this. We will describe this logic by giving a set of category-theoretical axioms, a now well-accepted approach to logic. There is a natural formal language that is entailed by these axioms (a form of dependent type theory), which we will describe only informally here; the full formal treatment will await another paper. The immediate payoff of all this is that we will give a sound foundation for the theory of categories over a base topos [P-S] [Be2] , a subject that has been around for a few years, but which is still lacking a formal backbone. In particular we will be able to express a weak form of the axiom of choice that will allow us to internalize all the classical constructions of category theory to a topos. This points the way towards the elimination of Grothendieck universes from the foundations of mathematics. Another, unexpected dividend of our methods is that they broaden and generalize our knowledge of the relation between formal systems and categorical properties; for example they allow us to give the category theoretical counterpart of less standard logics, like existential fixed point logic.

The logic that we want to describe departs from traditional type theory in two essential points. The first one is that we make the concept of *groupoid* (i.e. a category all whose morphisms are isos), instead of that of set, the notion whose essence we want to capture axiomatically. In particular, we will give an abstract categorical framemork which has the same role towards the 2-category of groupoids as the notion of an elementary topos has for the category of sets. The second point (a consequence of the first one) is that the "natural" logic that we will be able to interpret in our categories uses (and in fact cannot do without) dependent type theory. This formalism, originally due to P. Martin-Löf, has been getting attention lately because of its applications in theoretical computer science. Let us explain why these two departures are necessary. But first we will give a quick review on how to interpret logic in an elementary topos.

We will fix once and for all a category *Set* which we will call the small sets. For simplicity we will assume *Set* is a model of Zermelo-Frankel with the axiom of choice; we do use choice rather often, and so "our own universe" is quite classical and Platonic. *Set* may contain a Grothendieck universe itself; in other words there may exist very small sets. We will use the terms "small" and "locally small" for categories in the obvious sense. When we just say "category" without qualifiers, questions of size are irrelevant. We denote composition in category by fg or $f \circ g$, according to readability; we use the classical, functional order, i.e. $\mathrm{cod}(g) = \mathrm{dom}(f)$. The morphisms of a category are also called arrows and maps, for variety. Our notation for applying a functor F to an object A or map f is usually FA, Ff but sometimes we will write $F(A)$, $F(f)$ instead.

Let $\mathcal{E}$ be a small elementary topos. The first step in interpreting logic in $\mathcal{E}$ is to assign, to every object X of $\mathcal{E}$, a type, which we will also denote X throughout this work. Every type has a countable collection of variables; we write $x{:}X$ to say " x is of type X ". For every finite list $X_1, \ldots, X_n$ of objects and every subobject $S \hookrightarrow X_1 \times \ldots \times X_n$ we assign an n-place predicate symbol $p_S(-,\ldots,-)$ of the obvious arity. This assignment is unique in the sense that between two different choices of a product diagram there is a unique canonical way to go from one to the other (we will not bother with function symbols for the time being.) A formula of typed first-order logic $\varphi(x_1,\ldots,x_n)$ (with x_i of type X_i) is interpreted as a subobject $[\![\varphi]\!] \hookrightarrow X_1 \times \ldots \times X_n$. There is some well-known technical trivia here we will not bother with, for example what to do with variables that do not appear free in φ. The interpretation of φ is defined by induction and to every connective of logic corresponds an operation on subobjects. For example conjunction is the intersection of subobjects, disjunction the binary sup. Quantification, say universal, is adjoint to the substitution (pullback) functor: the subobject $[\![\forall_x \varphi(x,y)]\!] \hookrightarrow Y$ is obtained by applying to $[\![\varphi(x,y)]\!]$ the right adjoint to the functor "pull back by π ": $\mathit{Sub}(Y) \to \mathit{Sub}(X \times Y)$, where $\pi{:}X \times Y \to Y$ is the projection. The difference between first order and higher order logic is contained in the types we can construct: the reason higher order logic can be interpreted in any topos is that given an object-type X the topos structure gives us access to $\mathbb{P}(X)$, its powerset type. Another higher-order type constructor, which can be defined in terms of $\mathbb{P}(-)$, is the function-space constructor: given objects X, Y, then there is a type $X \Rightarrow Y$ of all functions of X into Y. The formal machinery associated to this type constructor is known as the simple (typed) lambda calculus.

A very useful tool for doing calculations in $\mathcal{E}$ is Kripke-Joyal semantics. Let $\varphi(x_1,\ldots,x_n)$ be a formula of the language of $\mathcal{E}$, whose interpretation is a subobject $[\![\varphi]\!] \hookrightarrow X = X_1 \times \ldots \times X_n$. The idea behind Kripke-Joyal is that the subobject $[\![\varphi]\!]$ is completely determined by the subfunctor of the contravariant representable $h_X = \mathcal{E}(-,X)$ (also called an X-sieve) it determines: that is, one can define the set

$$[\![\varphi]\!]^{\#} = \{(I,a) \mid a: I \to X \text{ and } a \text{ factors through } [\![\varphi]\!]\}$$

as the projection in our world of the "internal subset" of X that φ defines. $[\![\varphi]\!]^{\#}$ has the obvious structure of being downwards closed: if $(I,a) \in [\![\varphi]\!]^{\#}$ and $f: J \to I$ then $(J,af) \in [\![\varphi]\!]^{\#}$; this is another way of saying that $[\![\varphi]\!]^{\#}$ can be viewed as a presheaf. A pair (I,a), $a: I \to X$ is called "an element of X of base I", and since it can be thought of as an I-indexed family $(a_i)_{i \in I}$ of elements of X, we have the slogan "to get hold of the internal set $\{x \in X \mid \varphi(x)\}$ one has to look at all families indexed by all objects of $\mathcal{E}$, and not only at the one-element families $1 \to X$". In particular the internal set X itself can be realized in our world as the set of objects of the slice category $\mathcal{E}/X$. All the connectives of logic have an interpretation this way, which we will recall: let φ be a formula, whose vector of free variables has collective type X, and let $a: I \to X$. Then,

- if φ is atomic: $a \in [\![\varphi]\!]^{\#}$ iff a factors through the subobject associated to φ.
- if $\varphi = \psi \Rightarrow \theta$: $a \in [\![\varphi]\!]^{\#}$ iff for all $J \in \mathcal{E}$, $f: J \to I$, if $af \in [\![\psi]\!]^{\#}$ then $af \in [\![\theta]\!]^{\#}$.
- if $\varphi = \psi \wedge \theta$: $a \in [\![\varphi]\!]^{\#}$ iff $a \in [\![\psi]\!]^{\#}$ and $a \in [\![\theta]\!]^{\#}$.

- if $\varphi = \psi \vee \theta$ $\quad$ $a \in [\![\varphi]\!]^{\#}$ iff there exists a (finite) covering $(f_j : J_j \to I)_{1 \le j \le n}$ such that $af_j \in [\![\psi]\!]^{\#}$ or $af_j \in [\![\theta]\!]^{\#}$ for all j.
- if $\varphi = \mathsf{T}$ $\quad$ $a \in [\![\varphi]\!]^{\#}$ always
- if $\varphi = \mathsf{F}$ $\quad$ $a \in [\![\varphi]\!]^{\#}$ iff I admits an empty cover, i.e. is the initial object.
- if $\varphi = (\forall_{y:Y})\psi$ $\quad$ $a \in [\![\varphi]\!]^{\#}$ iff for all $J \in \mathcal{E}$, $f : J \to I$, $b : J \to Y$, $\langle b, af \rangle \in [\![\psi]\!]^{\#}$, where $\langle b, af \rangle : J \to Y \times X$.
- if $\varphi = (\exists_{y:Y})\psi$ $\quad$ $a \in [\![\varphi]\!]^{\#}$ iff there exists a (finite) covering $(f_j : J_j \to I)_{1 \le j \le n}$, and $(b_j : J_j \to Y)_{1 \le j \le n}$ such that $\langle b_j, af_j \rangle \in [\![\psi]\!]^{\#}$ for all j.

Notice that the definition above is valid for any subcanonical site, as soon as atomic formulas are assigned to subobjects; some minor simplifications could be given when $\mathcal{E}$ is a topos. An essential property of the definition above for a general site is that it conserves sheafness; thus if atomic formulas are assigned to sheaves (which is always the case if the site is subcanonical), $[\![\psi]\!]^{\#}$ is guaranteed to be a sheaf.

The point of this is that in an elementary topos Kripke-Joyal semantics for the topology of finite coverings coincides with the ordinary semantics based on universal operations on subobjects [L-S, 8.4]. One way to look at the relation between the two semantics is to say that the finite coverings site on an elementary topos obeys a nice *comprehension scheme*: if atomic relation symbols are assigned to every subobject of the topos, then for any first-order formula (meaning that no type of the form $\mathcal{P}(X)$ appears) of that language the sieve that models it is representable as a subobject.

Thus, it would only be natural to define a class inside a topos $\mathcal{E}$ as a a sheaf $H : \mathcal{E}^{op} \to \mathit{Set}$ for the finite covering topology, or at the very least, as a presheaf. For every $I \in \mathcal{E}$ the set HI would be thought of as the set of all I-indexed families of elements of H. Given $a \in HI$ (i.e. $(a_i)_{i \in I}$) and $f : J \to I$ then $(Hf)(a) \in HJ$ would be the family obtained by reindexing, i.e. $(a_{f(j)})_{j \in J}$. H would be said to be "small" if it were representable. What should H be if it were to correspond to the class of all objects of $\mathcal{E}$? Well, given $I \in \mathcal{E}$ category theorists have known for a long time how to model an I-indexed family $(E_i)_{i \in I}$ of objects of $\mathcal{E}$: an element of HI should be just an object $e:E \to I$ of $\mathcal{E}/I$; E is to be thought of as the disjoint sum $\Sigma_i E_i$. Given a morphism $f:J \to I$ then the operation of reindexing should be modelled by pullback. This is where our troubles begin: given an average topos, there is no way to define pullback *functorially*, since pullback is defined up to unique isomorphism, and not identity. One way out of this problem is to add the necessary structure to $\mathcal{E}$ to make pullbacks functorial: making $\mathcal{E}$ a τ-category [Fr-Sc] will fit the bill. It can be shown [Fr-Sc WW] that any category with finite limits is equivalent to a τ-category. But this goes against the grain of the whole tradition of category theory. Suddenly we are dealing with properties of categories that are not maintained under equivalence. Also, making pullbacks functorial seems artificial, since the categories we meet in practice are not that way usually (there is an important exception to this: in many of the categories that one meets in dependent type theory, e.g. syntactical categories, some classes of arrows admit functorial pullbacks.) It turns out that our semantics can be applied to τ-categories anyway, so we will get the best of both worlds.

So let us give up the idea of considering $\mathcal{E}$ as a presheaf over itself. What is the next best thing? Well, pullback being defined up to isomorphism, maybe we should include the isomorphisms in the structure we want to define: that is, given $I \in \mathcal{E}$ HI could be defined to be the *groupoid* with objects all maps $E \to I$ and arrows isomorphisms between maps: HI is the underlying groupoid of $\mathcal{E}/I$. That way pullback becomes a pseudofunctor $\mathcal{E}^{op} \to \mathit{Gpd}$, something which is functorial only up to (coherent) isomorphism. We think pseudofunctors are best approached via Grothendieck fibrations [WWW] , and we will use that language from now on. Thus H is the codomain functor $d_1: \mathcal{E}_{pb} \to \mathcal{E}$ where $\mathcal{E}_{pb}$ is the subcategory of the arrow category $\mathcal{E}^{\to}$ with all its objects but where a morphism is a pullback square. Therefore the first step of our program is to replace presheaves by fibrations of groupoids: groupoids are the next best thing to sets.

1.1 Proposition

Let $\mathbf{X}:\mathbb{X} \to \mathbb{C}$ be a functor between two categories. The following are equivalent:

i) **X** is a Grothendieck fibration such that for any $I \in \mathbb{C}$ the fiber $\mathbb{X}^I$ is a groupoid (it is a fibration of groupoids).

ii) **X** is a Grothendieck fibration for which all the morphisms are cartesian.

iii) **X** is a Grothendieck fibration and reflects isos.

The proof is trivial. Let $Fib_G/\mathcal{E}$ denote the category whose objects are the fibrations of groupoids over the topos $\mathcal{E}$ and the morphisms commuting triangles. For the time being let us consider the objects of this category as the classes we want to define. In particular, given $\mathbf{X} \in Fib_G/\mathcal{E}$ and $I \in \mathcal{E}$ we will think of the fiber above I as the groupoid of all I-indexed families of objects of **X** . There is an obvious embedding $\mathit{Set}^{\mathcal{E}^{op}} \longrightarrow Fib_G/\mathcal{E}$, the Grothendieck construction for presheaves. In particular we can compose with the Yoneda embedding and consider the objects of $\mathcal{E}$ as objects of $Fib_G/\mathcal{E}$. $\mathit{Set}^{\mathcal{E}^{op}}$ has the structure of a topos, and is thus a very good candidate for a universe of all classes; since it cannot do the job for us we have to ask ourselves if we can find some similar, topos-like structure (something allowing us to interpret a useful form of logic, at least) in $Fib_G/\mathcal{E}$. Just as the the objects of $\mathit{Set}^{\mathcal{E}^{op}}$ are often thought of as "$\mathcal{E}$-moving sets", those of $Fib_G/\mathcal{E}$ can be considered as "$\mathcal{E}$-moving groupoids". There is another possible notion of "$\mathcal{E}$-moving groupoid", the category $\mathit{Gpd}^{\mathcal{E}^{op}}$ of all presheaves of groupoids; this is also the category of all groupoid objects in the topos $\mathit{Set}^{\mathcal{E}^{op}}$, and we know it is too "strict" for us; so what we are studying could be called "loosely $\mathcal{E}$-moving groupoids". In going from $\mathit{Set}^{\mathcal{E}^{op}}$ to $Fib_G/\mathcal{E}$ there is some obvious added structure we will have to take into account: if $\mathbf{X}:\mathbb{X}\to\mathcal{E}$, $\mathbf{Y}:\mathbb{Y}\to\mathcal{E}$ are objects of $Fib_G/\mathcal{E}$ and $F,G:\mathbf{X}\to\mathbf{Y}$ morphisms there might exist a nontrivial natural transformation $F\to G$ between these "$\mathcal{E}$-moving functors" , it being an ordinary natural transformations $\alpha:F\to G$ satisfying the condition $\mathbf{Y}\alpha=\mathbf{X}$ (since all morphisms in $\mathbb{Y}$ are cartesian all morphisms of fibrations will be cartesian.) Thus we will working inside a 2-category, all whose 2-cells are isomorphism (also called a groupoid-enriched category.) This is where we depart from ordinary logic: the types (or objects) of our logic have to be thought of as groupoids, sure. But we, as set-trained mathematicians, always think of a groupoid as a set of objects (well, a class maybe) with added structure. Every groupoid in our own set-theoretical world has an underlying set/class. But this is not possible here: given $\mathbf{X}:\mathbb{X}\to\mathcal{E}$ in $\mathrm{Fib}_G/\mathcal{E}$ there is no way to guarantee that there is a subcategory $\mathbb{X}_0\subset\mathbb{X}$ with the same objects as $\mathbb{X}$ such that by composing we get a discrete fibration $\mathbb{X}_0\to\mathcal{E}$. We will have to change our "setist" point of view and think the other way around: take the concept of groupoid as the primitive notion. Thus a set is a special kind of groupoid, a discrete one. In other words this universe is such that given a groupoid that lives in it it is not necessarily possible to extract the objects and morphisms as independent structure. Things can be tightly bound.

This does have some philosophical bearing on the foundations of mathematics. The theory of sets is not the end of the problem of foundations: for example, there have been little success in formalizing the notion of *mathematical structure*, which is truly at the center of modern mathematics. We know how do deal with given classes of structures (say varieties, or elementary classes), and there is a given such class (all structures definable with higher-order logic) which seems to be large enough to encompass all constructions encountered in "ordinary mathematics with small sets". But it is still very ad hoc, since for example we are bound by the choice of function symbols, atomic predicates, etc whenever we want to describe a class of structures.

In a foundation of mathematics based on the idea of structure the notion of groupoid would be central: given a class of structures, say topological spaces, the one given notion of morphism that is guaranteed to be invariant is that of isomorphism. Continuous functions have their use, and so do open functions, or local homeomorphisms, but none of these definitions follows immediately from the definition of a space itself. But given only the definition of a topological space it is obvious how to tell a homeomorphism: it is a correspondence between two spaces that are "the same" space. Every mathemetician knows by experience that isomorphism is a much better notion of "sameness" between two mathematical structures than actual equality (which is hard to define in general and very bound to conventions of notation and coding). Category theorists are willing to say more, and that the actual notion of equality between structures (here meaning between objects of a category) is a dangerous notion, since use of the equality-between-objects symbol in formulas is not invariant under equivalence of categories [Fr] [Bl].

So let us think of an object $\mathbf{X}:\mathbb{X}\rightarrow\mathcal{E}$ of $\mathrm{Fib}_G/\mathcal{E}$ as a "class of structures in $\mathcal{E}$, along with all isomorphisms". We want to interpret logic in there, and a predicate $\varphi(x)$, $x:\mathbf{X}$ will be interpreted as a subobject of $\mathbf{X}$. But if we are consistent with ourselves we will not accept any kind of subobject. It is natural to ask that the property $\varphi(x)$ be closed under isomorphisms: if $\varphi(a)$ and a' is isomorphic to a then $\varphi(a')$ too. This is just saying that $\varphi(a)$ is a property intrinsic to the structure a. This translates in $\mathrm{Fib}_G/\mathcal{E}$ by saying that $[\![\varphi]\!]\hookrightarrow\mathbf{X}$ will have the form

$$\begin{array}{ccc} \mathbb{Y} & \hookrightarrow & \mathbb{X} \\ {[\![\varphi]\!]} \searrow & & \swarrow \mathbf{X} \\ & \mathcal{E} & \end{array}$$

of a diagram of fibrations where $\mathbb{Y}\hookrightarrow\mathbb{X}$ is the inclusion of a full subcategory and is closed under isomorphism classes (a rather standard terminology is to say that the subobject is <u>replete</u>.) Such a condition can be interpreted in any 2-category as we will soon see. Then an interesting phenomenon happens: the diagonal $\mathbf{X}\hookrightarrow\mathbf{X}\times\mathbf{X}$ is not replete unless $\mathbf{X}$ is a discrete fibration, as the reader can easily check, remembering that the product in $\mathrm{Fib}_G/\mathcal{E}$ is given by pullback of fibrations. In other words equality between objects is not a predicate of this logic of groupoids, and this is consistent with our philosophy. If the reader thinks that this philosophy blinds us to other possible approaches there is another, purely technical reason that forces us to interpret predicates by replete subobjects: a category of fibrations over a base category does not have pullbacks in general, and pullbacks are essential for interpreting substitution. One can easily construct examples where the interpretation of $F(x)=G(x')$, for $F,G:\mathbf{X}\rightarrow\mathbf{Y}$ is impossible, because the equalizer of F, G (which can be constructed as the pullback of the diagonal on $\mathbf{Y}$) does not exists. J. Penon [Pe] has proposed the use of "bi-categorical" properties to remedy this situation, that is, requiring that the ordinary notion of pullback be replaced by a looser form, defined up to equivalence and no up to isomorphism. Let us state our position here about these matters: "bi-properties" cannot be avoided, as we will see, but their use should be restricted as much as is possible, since they are complicated and do not merge well with ordinary syntax. The ultimate goal is to make "bi-properties" invisible by the means of a coherence theorem.

One detail has to be cleared before we start discussing formal issues: so far we have used the 2-category $\mathrm{Fib}_G/\mathcal{E}$ as a paradigm, and neglected the gluing conditions that should arise if one wants to mimic sheaves and not just presheaves; for example, if $H: \mathbb{H} \to \mathcal{E}$ is an $\mathcal{E}$-class and $f: J \to I$ an epi of $\mathcal{E}$, $a \in H^J$ (that is, informally, a is a family $(a_j)_{j \in J}$ of objects of H) such that "whenever $f(j) = f(j')$ then a_j and $a_{j'}$ are the same" then there should be $b \in H^I$ such that " a_j is the same as $b_{f(j)}$ ". The parallel to a sheaf in the world of fibrations is called a stack [WWW]. The definition of a stack is rather more technical than that of a sheaf due to the fact that one has to correctly handle the concept of "the same" in this looser context. The isomorphisms used for doing so have to satisfy some coherence relations; the reader should note that a sheaf of groupoids is not necessarily a stack, but a stack of discrete groupoids is a sheaf and vice versa. Things turn out for us as they should:

1.2 Theorem [???]

Let $\mathcal{E}$ be a pretopos. Then the fibration codomain: $\mathcal{E}_{pb} \to \mathcal{E}$ where $\mathcal{E}_{pb}$ is the subcategory of $\mathcal{E}^{\to}$ having the same objects and pullback squares for morphisms is a stack for the topology of finite coverings.

The category of stacks over $\mathcal{E}$, for the finite covering topology, will be our most desirable universe of classes for doing category theory in a topos. In particular the first-order logic of the topos (along with the topos' lambda calculus, but not Ω) is embedded conservatively in this universe. But along the way we will discover other interesting universes, whose categories of small sets may be much weaker than toposes.

So we are on our way to developing something which can be called "the natural logic of groupoids", and which we claim is "the natural logic of categories". The advantage of this form of logic is that the properties of categories it will allow us to express are invariant under equivalence type, quite a desirable feature as has been said before. Formal systems of that sort have been proposed [Fr]; what we are presenting here is as close as possible, we think, to the tradition of ordinary logic with variables and quantifiers.

We can now explain why the elimination of equality between objects forces us to resort to the use of dependent types. Let us go back to interpreting logic in the topos $\mathcal{E}$ and give an example in there: let $f{:}Y \to X$ be a morphism of $\mathcal{E}$. How do we express internally that f is surjective? Well, we just write

$$\forall_{x:X} \exists_{y:Y} f(y) = x \qquad .$$

By the miracle of topos semantics, this formula is true in $\mathcal{E}$ iff f is a regular epimorphism, in other words a one-element covering family of X. But we had to use the equality symbol. Is there a way it can be eliminated? Suppose now f is to be seen as an indexed family, and that we can write it as $(Y_x)_{x:X}$, with the intuition "$Y_x = f^{-1}(x)$". Then we could simply express surjectivity by

$$\forall_{x:X} \exists_{y:Y_x} \top$$

(The instance of $\top$ is there because we need a predicate to apply the quantifiers to!) The use of indexed families, if we can formalize it, will allow us to eliminate many instances of the equality symbol.

This formal theory of indexed families exists, and is called dependent type theory; it is mainly the brainchild of P. Martin-Löf. The idea is that a type may have variables in it, to express the idea of a family of types, indexed by (or depending on) these variables. The actual detailed syntax is rather ponderous, but its use in an application like ours (where no computer is involved) is quite simple and natural, since it corresponds to common mathematical practice. The complications come from the fact that a new variable may depend on other, previously defined variables. This precludes the standard rule "for every type there is a countable set of variables of that type". Instead the variables are typeless; they are given a type "upon appearance", in what is called a type declaration. In other words the formal syntax has entities of the form

$$x_0{:}X_0\ ,\ x_1{:}X_1\ ,\ x_2{:}X_2\ \ldots,\ x_{n-1}{:}X_{n-1} \qquad , \quad (*)$$

called <u>contexts</u>, where $x_i{:}X_i$ means that x_i is a variable which is declared to be of type X_i; a context is subject to the obvious condition that the only variables that may appear free in X_i are $x_0, \ldots, x_{i-1}$. In particular, X_0 is always guaranteed to be an ordinary, independent type, and thus will be modelled by an object $\mathbf{X}_0$ of the category. Let us assume for the time being that the category in which we are doing semantics is the topos $\mathcal{E}$. As is to be expected the dependent type $X_1(x_0)$, with only one free variable, is to be modelled by a morphism $X_1:\mathbf{X}_1\to\mathbf{X}_0$ of $\mathcal{E}$. If x_0 does not appear free in X_1, this means that X_1 is an independent type too, and so there has to be an object $\mathbf{Y}$ such that X_1 is the projection $\mathbf{X}_0\times\mathbf{Y}\to\mathbf{X}_0$. The type $X_2(x_0,x_1)$ is modelled by a morphism $X_2:\mathbf{X}_2\to\mathbf{X}_1$, and so on: the full context $x_0:X_0,\dots,x_n:X_n$ is interpreted as a sequence

$$\mathbf{1}\xleftarrow{X_0}\mathbf{X}_0\xleftarrow{X_1}\mathbf{X}_1\dots\xleftarrow{X_{n-1}}\mathbf{X}_{n-1}\xleftarrow{X_n}\mathbf{X}_n \quad , \quad (**)$$

of morphisms of $\mathcal{E}$.

Suppose that the context

$$x:X\ ,\ y:Y(x)$$

is interpreted by the morphism $Y:\mathbf{Y}\to\mathbf{X}$, as above. Let $S\hookrightarrow\mathbf{Y}$ be a subobject. It is natural to assign to it an atomic predicate $p_S(x,y)$; this would read in ordinary notation as an X-indexed family of predicates $p_{S,x}(y)$ over Y_x. We will write this as

$$[\,x:X\,,y:Y(x)\,]_p\ p_S(x,y) \quad ,$$

meaning "given the assignments of variables in the context the expression to the right of the brackets is a well-formed predicate". Such an expression is called a <u>predicate judgement</u>. Now we know that the pullback functor $Y^*:\mathit{Sub}(\mathbf{X})\to\mathit{Sub}(\mathbf{Y})$ has a right adjoint. Applying this adjoint to S gives a subobject of $\mathbf{X}$; its syntactical counterpart is the judgement

$$[\,x:X\,]_p\ \forall_{y:Y(x)}\,p_S(x,y) \quad .$$

Since y is no longer free it can be removed from the context: its typing information has been transferred to the predicate. The same obviously can be done with $\exists$. This gives an example of how judgements allow us to formalize bounded dependent quantification. In general, a predicate judgement

$$[\,x_0:X_0,\dots,x_n:X_n\,]_p\ \varphi$$

tells us that we have an interpretation $[\![\varphi]\!]\hookrightarrow\mathbf{X}_n$ for φ and gives us the right (at least in a topos) to form the judgements
$[\,x_0:X_0,\dots,x_{n-1}:X_{n-1}\,]_p\ \forall_{x_n:X_n}\,\varphi$ and
$[\,x_0:X_0,\dots,x_{n-1}:X_{n-1}\,]_p\ \exists_{x_n:X_n}\,\varphi$, by applying the appropriate adjoint

to the pullback functor X_n^*. This means that the natural dependent type theory of toposes has two *predicate formation rules*:

$$\frac{[\Gamma, y{:}Y]_p\ \varphi}{[\Gamma]_p\ \forall_{y:Y}\ \varphi} \qquad \frac{[\Gamma, y{:}Y]_p\ \varphi}{[\Gamma]_p\ \exists_{y:Y}\ \varphi},$$

where Γ stands for a list of type declarations $x_i{:}X_i$.

If we wanted to formalize the language in full detail we would have to explicitely mention a predicate formation rules like

$$\frac{[\Gamma]_p\ \varphi \quad [\Gamma]_p\ \psi}{[\Gamma]_p\ \varphi \wedge \psi}$$

for every logical connector. And there is more: the method of judgements has to be applied not only to predicates, but also to types. Remember the well-know theorem

> Given a morphism $f{:}I \to J$ in a topos $\mathcal{E}$ the pullback functor $f^*{:}\,\mathcal{E}/J \to \mathcal{E}/I$ has a both a right and a left ajoint, denoted Π_f and Σ_f respectively.

This suggests two type-forming operations. First, let us introduce another kind of judgement, called a *type judgement*, whose brackets are not decorated:

$$[x_0{:}X_0, \dots, x_n{:}X_n]\ A \qquad .$$

Now A is a type in which only $x_0, \dots, x_n$ are allowed to appear free. Its meaning is "A is a well-formed type that depends on the variables assigned in the context". Its interpretation will be a chain of morphisms

$$\mathbf{1} \xleftarrow{X_0} \mathbf{X}_0 \xleftarrow{X_1} \mathbf{X}_1 \dots \xleftarrow{X_{n-1}} \mathbf{X}_{n-1} \xleftarrow{X_n} \mathbf{X}_n \xleftarrow{A} \mathbf{A} \quad .$$

Then, in a topos, given a type judgement as the one above, one has the right to construct the judgements $[x_0{:}X_0, \dots, x_{n-1}{:}X_{n-1}]\ \Pi_{x_n:X_n} A$ and $[x_0{:}X_0, \dots, x_{n-1}{:}X_{n-1}]\ \Sigma_{x_n:X_n} A$. The interpretation of $\Pi_{x_n:X_n} A$ in $\mathcal{E}$ is the morphism $\mathbf{B} \to \mathbf{X}_{n-1}$ obtained by applying the right adjoint to $X_0^*{:}\,\mathcal{E}/\mathbf{X}_{n-1} \to \mathcal{E}/\mathbf{X}_n$ to A. The interpretation of $\Sigma_{x_n:X_n} A$ is the same, using the left adjoint instead; we know it is just the composite $X_nA{:}\,\mathbf{A} \to \mathbf{X}_{n-1}$. These constructions have been used a lot in topos theory, since dependent products and sums are a standard mathematical construction, but their correct syntactical framework has not been well known at all among category theorists. Π and Σ are binding operators, like ordinary quantifiers.

Notice that in a judgement like $[x_0{:}X_0, \ldots, x_n{:}X_n]X_{n+1}$ there is an element of arbitrariness since the order on variables (and types) is linear, due to typographical constraints. There is a natural suborder $x_i \prec x_j$ of that total ordering on the variables, that expresse the dependency of x_j on x_i : write $x_i \prec x_j$ for the transitive closure of the relation " x_i appears free in X_j ". Let us add a dummy variable x_{n+1} to take account of X_{n+1} in the order. This is obviously a transitive, antireflexive, antisymetric relation, i.e. an ordering without reflexivity. We will enforce a notational convention by requiring that *only the* $x_i \prec x_j$ *that are* $\prec$*-maximal should appear free in* X_j . The point is that in a judgement like $[x{:}X, y{:}Y(x)]_p\, ps(x,y)$ the presence of x in ps is not necessary, since x appears in the context already and y cannot exist without x anyway. So our practice will differ from what we have done so far, and we will write this sample judgement $[x{:}X, y{:}Y(x)]_p\, ps(y)$ instead. In ordinary, informal mathematical practice, people will also use more fastidious, redundant notations along with this approach, for reasons of emphasis. Let us end this preliminary discussion by saying how terms will be handled: there will be another kind of judgement, called a <u>term judgement</u> that has the form

$$[x_0{:}X_0, \ldots, x_n{:}X_n]\, f{:}A$$

and whose meaning will be " f is a well-formed term of type A ", and whose interpretation will be a diagram

$$\begin{array}{cccccc} X_0 & X_1 \;\cdots & X_{n-1} & X_n & A & \\ \mathbf{1} \longleftarrow \mathbf{X}_0 \longleftarrow & \mathbf{X}_1 \ldots \longleftarrow & \mathbf{X}_{n-1} \longleftarrow & \mathbf{X}_n & \underset{f}{\overset{\longleftarrow}{\longrightarrow}} \mathbf{A} & \end{array}$$

where f is a section of A , i.e. $Af = 1_{\mathbf{X}_n}$. The same constraint as for types will apply for the free variables in f , that is, they will always be maximal for the $\prec$-order.

§2 THE FORMALISM OF DEPENDENT TYPE THOERY

Since we want to work in more general contexts than elementary toposes let us start by giving what is almost (i.e. cf. [Eh]) the most general axiomatic context that allows one to do dependent type theory. And general it is:

2.1 Definition

A display category [H-P,Ca,???] $(\mathfrak{C},\mathfrak{F})$ consists of a category $\mathfrak{C}$ with a terminal object, along with a distinguished class $\mathfrak{F}$ of arrows, the display maps , or abstract fibrations such that

i) If $E:\mathbf{E}\to\mathbf{C}$ is a display map and $F:\mathbf{D}\to\mathbf{C}$ any arrow then the pullback $F^*E:F^*\mathbf{E}\to\mathbf{D}$ exists and is a display map

ii) If E is as above and $\alpha:\mathbf{C}\to\mathbf{C}'$, $\beta:\mathbf{E}'\to\mathbf{E}$ isomorphisms then $\alpha E\beta$ is a display map.

A display category is said to be common if it obeys the additional condition

iii) every morphism to the terminal object is a display map.

The display maps correspond to dependent types and are the morphisms that will appear in the interpretations of contexts and judgements; in other words not all morphisms of a category need have meaning as a type (we have already seen this happen in our discussion of pullbacks of replete monos in categories of fibrations.) Condition i) is called Stability in [H-P] , and says that we can substitute any term of the right type for a variable in a dependent type. Condition ii) is a natural complement to condition i) , and comes from our desire of making things as close as possible to categorical practice. It says that the property of being a display map is intrinsic to the map, and not an arbitrary whim. iii) says that any object can be seen as an independent type. Since there is a terminal object, we have a type for the one-element set. If $\alpha:\mathbf{X}'\to\mathbf{X}$ is an isomorphism, the square

$$\begin{array}{ccc} \mathbf{X}' & \longrightarrow & \mathbf{1} \\ \alpha\downarrow & & \downarrow \\ \mathbf{X} & \longrightarrow & \mathbf{1} \end{array} \quad .$$

is a pullback and the identity of the terminator is a display map by iii) , so we get that in a common display category every iso is a display map. Another consequence of i) and iii) is that $\mathcal{C}$ has finite products and every projection is a display map, since taking the product is pulling back maps to the terminator.

The assignment of a language to a display category will proceed in two steps: the first step is for all atomic symbols and will be described in more details than the second step, which is for the composite types and terms, those that are built using formation rules and substitution (the reader can skip 2.2-2.4 at first reading). In order to carry out the first step, we have to use the axiom of choice and designate a pullback diagram

$$\begin{array}{ccc} & X^{\dagger}Y & \\ \cdot & \longleftarrow & \cdot \\ Y\downarrow & & \downarrow X^{\dagger}Y \\ \cdot & \longleftarrow & \cdot \\ & X & \end{array}$$

for every pair of display maps with a common codomain. If X is an identity morphism we ask that $X^{\dagger}Y = Y$. This choice of pullbacks does not have to be functorial, which differs from [St] [Ca] . We will also need to choose the terminal object once and for all, and we will call it **1** . Recall that a <u>graph</u> G is a pair of sets (G_0,G_1) (the <u>vertices</u> and the <u>edges</u> , respectively) along with a pair of functions $s,t:G_1 \rightarrow G_0$, called <u>source</u> and <u>target</u> . A <u>path</u> $\alpha_0,\ldots,\alpha_n$ in G is a sequence of edges such that $s(\alpha_{i+1}) = t(\alpha_i)$.

2.2 Definition

A <u>template</u> is a pair (G,r) , where G is a finite graph (G_0,G_1,s,t) and r an endofunction $G_1 \rightarrow G_1$, called <u>reference</u>, subject to the following conditions:

i) G is a tree: for every vertex v there is a unique edge with source v, and there is a vertex $*$ such that for every $v \in G_0$ there is a path $\alpha_0, \dots, \alpha_n$ (necessarily unique) with $s(\alpha_0) = v$ and $t(\alpha_n) = *$. In particular the edge with source $*$ has also $*$ for target. This induces an order structure on G_1: we write $\alpha \leq \beta$ to mean that there is a path $\beta, \dots, \alpha$. This order is usually defined on G_0: say $v < v'$ if there is a path $\alpha_0, \dots, \alpha_n$ with $s(\alpha_0) = v'$ and $t(\alpha_n) = v$; we will use the orders on both G_0 and G_1.

ii) r is eventually idempotent, i.e. for every $\alpha \in G_1$ there is n such that $r^n(\alpha) = r^{n+1}(\alpha)$. Therefore every $\alpha \in G_1$ determines a unique fixpoint $f(\alpha)$.

iii) r is injective except on the fixpoints, i.e. $r(\alpha) = r(\beta) \Rightarrow \alpha = \beta = r(\alpha)$.

iv) for every edge α which is not a fixpoint $r(\alpha) \not\leq \alpha \not\leq r(\alpha)$ and $t(\alpha) \neq t(r(\alpha))$.

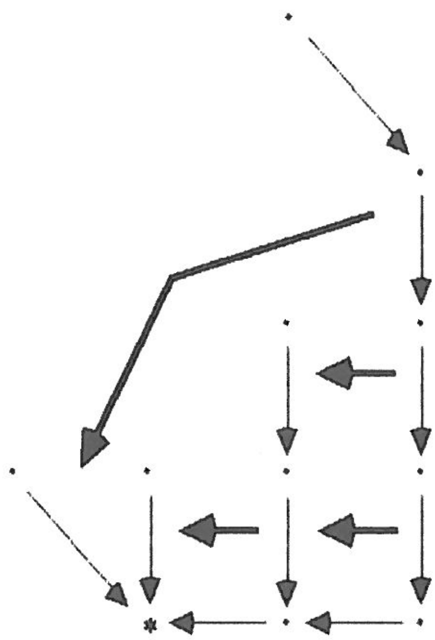
v) If $\beta = r(\alpha)$ and $t(\beta) \not< t(\alpha)$ then there exists $\gamma < \alpha$ such that $r(\gamma)$ is the predecessor of β, i.e. the unique maximal edge $< \beta$.

vi) There is a path, $\alpha_0, \dots, \alpha_n$, the <u>mainline</u>, where α_0 is maximal, such that all the edges not in the image of r are in the mainline, and all other mainline edges are fixpoints. This path is uniquely defined; in order for this to happen, it suffices to require that the set $G_1 - \mathrm{Im}(r)$ be linearly ordered and that all the edges between its elements and above its top element be fixpoints and form a path.

vii) Every edge not in the main line has a maximal fixpoint edge above it.

We say that a template is <u>prime</u> if the maximal edge of its mainline is a fixpoint. Let (G,r) be a template and $\alpha \in G_1$ be a fixpoint. There is a sub-template generated by α and it is prime: take the path between α and $*$ and let G_1' be the closure of this subset under the operation r. An easy verification shows that it is a template and its mainline is the path $\alpha, \dots, *$. It is also easy to see that given a prime template (G,r) and α the maximal mainline edge then the template generated by α is (G,r).

The reader should not panic and look at the examples to get an idea of what is going on. We should add that conditions iv) and vii) are there to normalize templates, that is, get a unique template to correspond to a given context, modulo the choice of variables. Here is a pictorial example of a template, which is obviously not prime: the thick arrows represent the action of r ; we do not show it for fixpoints. The mainline is obviously the only path of length six.



The intuitive meaning is as follows: all the edges of the template are to be interpreted as display maps in $\mathcal{C}$, and given a fixpoint arrow this map will also be used as an atomic type symbol in the syntax; the reference function says that an edge α is to be interpreted as the pullback of $r(\alpha)$, but that in the syntax its type symbol is not that pullback map, but the map corresponding to the fixpoint $f(\alpha)$. In order to make all this precise, recall that given a graph G a G-diagram (or diagram of base G) in $\mathcal{C}$ is a morphism of graphs from G to the underlying graph of $\mathcal{C}$. Given a diagram $D: G \rightarrow \mathcal{C}$ we will use the subscript notation for edges and vertices, i.e. if $\alpha: v \rightarrow v'$ in G then $D_\alpha: D_v \rightarrow D_{v'}$.

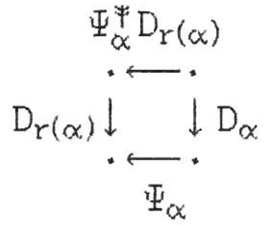
2.3 Definition

Let $T = (G,r)$ be a template. A decoration of T in $\mathcal{C}$ is a diagram $D: G \rightarrow \mathcal{C}$, along with a family Ψ of maps $(\Psi_\alpha: D_{t(\alpha)} \rightarrow D_{t(r(\alpha))})_{\alpha \in G_1}$ such that

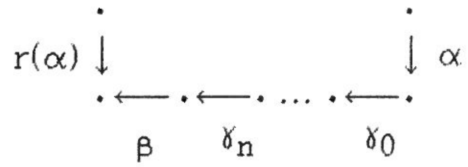
i) $D(*) = \mathbf{1}$.

ii) If α is a fixpoint Ψ_α is identity.

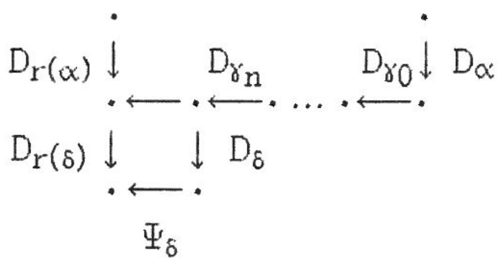
iii) For every α , $D_\alpha = \Psi_\alpha^\dagger D_{r(\alpha)}$.



iv) If α is not a fixpoint, then by iv) either $t(r(\alpha)) < t(\alpha)$ or $t(r(\alpha))$ is unrelated (neither $\leq$ nor $\geq$) to $t(\alpha)$. In the first case let β be the edge $< \alpha$ such that $t(\beta) = t(r(\alpha))$. Let $\alpha, \gamma_0, \ldots, \gamma_n, \beta$ be the unique possible path. Then $\Psi_\alpha = D_\beta \circ D_{\gamma_n} \circ \ldots \circ D_{\gamma_0}$.



In the second case, by v) there is $\delta < \alpha$ such that $r(\delta)$ is the predecessor of $r(\alpha)$. Let $\alpha, \gamma_0, \ldots, \gamma_n, \delta$ be the unique possible path. Then $\Psi_\alpha = \Psi_\delta^\dagger D_{r(\delta)} \circ D_{\gamma_n} \circ \ldots \circ D_{\gamma_0}$.

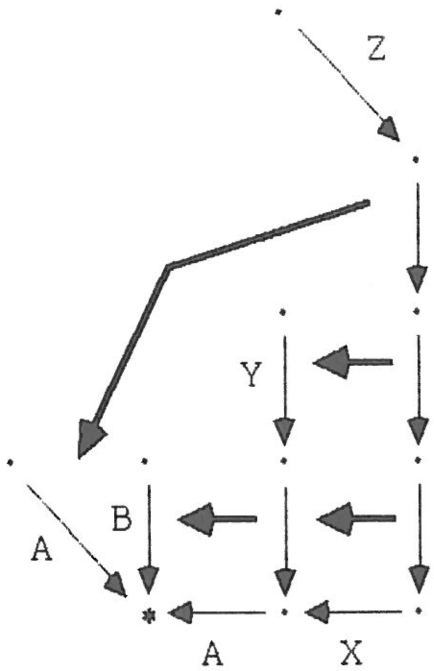


2.4 Proposition

A decoration $(D,(\Phi_\alpha)_\alpha)$ is entirely determined by the values D_α for all fixpoint edges α .

The proof is easy and done by induction on the length of the segment below α . □

For example if we assign display morphisms of $\mathbb{C}$ to the fixpoint edges of the template drawn above



we define a decoration if and only if $\mathrm{cod}(A) = \mathrm{cod}(B) = \mathbf{1}$, $\mathrm{cod}(X) = \mathrm{dom}(A)$, $\mathrm{cod}(Y) = \mathrm{dom}(A^{\dagger}B)$ and $\mathrm{cod}(Z) = (A \circ X \circ X^{\dagger}(A^{\dagger}B) \circ (X^{\ddagger}(A^{\dagger}B))^{\dagger} Y)^{\dagger} A$.

We can now associate a formal language to our display category. First, to all triples (T,D,Ψ) where T is a prime template and (D,Ψ) a decoration of T , associate an atomic type symbol $\langle D_{\alpha},T,D,\Psi \rangle$, α being the generator of T . In practice we will simply use the display map D_{α} to denote this type; the added information is to make type symbols impervious to the identifications of objects of $\mathbb{C}$ brought about by the chosen pullbacks. Thus we lied a little when we said we would associate atomic types to display maps; a bit more is needed. Let now $T = (G,r)$ be any template and (D,Ψ) a decoration of T . Let $\alpha_0, \ldots, \alpha_n$ be its mainline, and let $x_1, \ldots, x_n$ be a sequence of *distinct* variables. If β is a fixpoint edge of G let $T|_{\beta}$ denote the prime template that has β as a generator, and $(D|_{\beta},\Psi|_{\beta})$ the decoration obtained by restricting (D,Ψ) to $T|_{\beta}$. We define the order $<$ on the variables as the transitive closure of the relation

$$x_i < x_j \text{ if there exists } m \text{ such that } r^m(\alpha_i) < f(\alpha_j) .$$

Let X_i denote the type symbol $\langle D_{f(\alpha_i)}, T|_{f(\alpha_i)}, D|_{f(\alpha_i)}, \Psi|_{f(\alpha_i)} \rangle$ (we get kickbacks from the White Knight.) The decorated template (T,D,Ψ) gives rise to the context

$$x_0{:}X_0 , x_1{:}X_1(\kappa_1), \ldots , x_{n-1}{:}X_{n-1}(\kappa_{n-1}) , x_n{:}X_n(\kappa_n) ,$$

where κ_i is the sublist (for us list is just another word for a finite sequence) of the list $x_0,\dots,x_{i-1},x_{i-2}$ of variables that retains only the $x_j \prec x_i$ that are $\prec$-maximal. For example if we choose the sequence of variables a,x,b,y,a',z the context we get (using morphisms as type symbols) from our sample template is

$$a{:}A\,,x{:}X(a)\,,b{:}B\,,y{:}Y(b)\,,a'{:}A\,,z{:}Z(x,y,a')\ .$$

In the definition that follows whenever we mention the decorated template (T,D,Ψ) all its innards are denoted just as above. That is, its mainline has length n, its edges are denoted α_i, etc. .

2.5 Definition

A basic context of $\mathbb{C}$ is a context $[x_0{:}X_0,\dots,x_n{:}X_n]$ obtained from a decorated template (T,D,Ψ) and a sequence of variables by the process above. Its interpretation is the diagram of display maps

$$\mathbf{1} \xleftarrow{D_{\alpha 0}} \cdot \xleftarrow{D_{\alpha 1}} \cdot \ \dots\ \longleftarrow \cdot \xleftarrow{D_{\alpha n}} \cdot \qquad .$$

A basic type judgement is a judgement $[x_0{:}X_0,\dots,x_{n-1}{:}X_{n-1}]X_n$ obtained by taking a basic context, dropping the last variable and putting the last type in evidence, out of the context. Its interpretation is the same sequence of morphisms as for the context one started with. Now look at the sequence $\alpha_n, r(\alpha_n), r^2(\alpha_n),\dots,f(\alpha_n)$, assuming that $f(\alpha_n)=r^m(\alpha_n)$. There is a sequence of contiguous pullback squares

$$\begin{array}{ccccccc}
\cdot & \longrightarrow & \cdot & \longrightarrow & \cdot\ \dots\ \cdot & \longrightarrow & \cdot \\
D_{\alpha n}\downarrow & & \downarrow D_{r(\alpha_n)} & & \downarrow & & \downarrow D_{f(\alpha_n)} \quad (*) \\
\cdot & \xrightarrow[\Psi_{\alpha n}]{} & \cdot & \xrightarrow[\Psi_{r(\alpha n)}]{} & \cdot\ \dots\ \cdot & \xrightarrow[\Psi_{r^{m-1}(\alpha_n)}]{} & \cdot
\end{array}$$

To every $f{:}D_{t(\alpha_n)} \to D_{s(f(\alpha_n))}$ such that $D_{f(\alpha n)} \circ f = \Psi_{r^{m-1}(\alpha_n)} \circ \dots \circ \Psi_{r(\alpha_n)} \circ \Psi_{\alpha_n}$ assign an atomic term symbol $\rho_{f,T,D,\Psi}$, which in practice we will denote by the morphism f. Then the judgement $[x_0{:}X_0,\dots,x_{n-1}{:}X_{n-1}]f(\kappa){:}X_n$ is a basic term judgement of $\mathbb{C}$, where κ is the sublist of the sequence $x_{n-1},x_{n-2},\dots,x_0$ that retains the variables that are $\prec$-maximal. The interpretation of this judgement is the diagram

$$\mathbf{1} \xleftarrow{D_{\alpha 0}} \cdot \xleftarrow{D_{\alpha 1}} \cdot \ \dots\ \longleftarrow \cdot \underset{f'}{\overset{D_{\alpha n}}{\leftrightarrows}} \cdot \qquad ,$$

where f' is the section of D_{α_n} determined by f and the fact that the outer rectangle in (*) is a pullback. For example, if $\mathbf{A},\mathbf{B},\mathbf{C}$ are objects of $\mathcal{C}$ such that $\mathbf{A}:\mathbf{A}\to\mathbf{1}, \mathbf{B}:\mathbf{B}\to\mathbf{1}, \mathbf{C}:\mathbf{C}\to\mathbf{1}$ are display maps, and $f:\mathbf{A}\to\mathbf{B}$ $g:\mathrm{dom}(\mathbf{A}^{\dagger}\mathbf{B})\to\mathbf{C}$ morphisms of $\mathcal{C}$, then $[a:\mathbf{A}]f(a):\mathbf{B}$ and $[a:\mathbf{A}, b:\mathbf{B}]g(b,a):\mathbf{C}$ are term judgements of $\mathcal{C}$. So the reason that we go through some contortions to use f and not f' (as defined just above) as the term symbol is just to make things look like ordinary practice and thus read more easily. Finally if $S\hookrightarrow\mathbf{X}_n$ is a display subobject (that is, such that its representatives are display maps) assign the predicate symbol $p_{S,T,D,\Psi}$. Then $[x_0:X_0,\dots,x_n:X_n]_p\, p_S(\kappa)$ is a <u>basic predicate judgement</u> of $\mathcal{C}$, where κ is the sublist of the variables $x_n, x_{n-1},\dots,x_0$ of all the x_i that are $\prec$-maximal. The interpretation is the diagram

$$\mathbf{1}\xleftarrow{D_{\alpha_0}}\cdot\xleftarrow{D_{\alpha_1}}\cdot\ \dots\ \longleftarrow\cdot\xleftarrow{D_{\alpha_n}}\cdot\hookleftarrow S\ .$$

(We will not be bothered by the fact that a subobject is not really a morphism and so cannot appear in a diagram; more on this soon.) For example let D be a prime template, such that the display map D_{α_n} has the property that its diagonal (that is, the splitting Δ of $D_{\alpha_n}{}^{\dagger}D_{\alpha_n}$ such that $D_{\alpha_n}{}^{\ddagger}D_{\alpha_n}\circ\Delta$ is identity) is a display map.

$$\begin{array}{ccc} & D_{\alpha_n}{}^{\ddagger}D_{\alpha_n} & \\ \cdot & \longleftarrow & \cdot \\ D_{\alpha_n}\downarrow & & \downarrow D_{\alpha_n}{}^{\dagger}D_{\alpha_n} \\ \cdot & \longleftarrow & \cdot \\ & D_{\alpha_n} & \end{array}$$

Construct a new template T' by adding one edge β with target $t(\alpha_n)$, one edge α_{n+1} with target $s(\alpha_n)$, and such that $r(\alpha_{n+1})=\beta=r(\beta)$. There is an obvious extension (D',Ψ') of the decoration (D,Ψ) that sends β to D_{α_n}. Then if y is a new variable, $[x_0:X_0,\dots,x_n:X_n,y:X_n]_p\, x_n =_{X_n} y$ is a basic predicate judgement of $\mathcal{C}$. Usually we will not bothering indexing the equality symbol by its type when we use it.

2.6 Note

In theory type and term symbols force the order of introduction of the variables that appear in them. For example, if $[x:X, y:Y]Z(x,y)$ is a basic type judgement of $\mathcal{C}$ and a,b variables then the type $Z(a,b)$ can only appear in contexts and judgements preceded by a variable declaration $\ldots a:X \ldots b:Y \ldots$, and never $\ldots b:Y \ldots a:X \ldots$. This is because $X^{\dagger}Y \neq Y^{\dagger}X$ in general. In practice one can always compose with the necessary isomorphism in order to use a symbol in a context whose variables are not in the right order.

Basic contexts and judgements can be combined to form composite terms and types. From now on though, we will not make choices whenever we mention a universal construction like pullback. Thus we will revert to the traditional notation $F^*(-)$ to denote the pullback "functor". The additional morphisms we will construct in the interpretations will only be defined "up to unique isomorphism" and should be thought of as equivalence classes of morphisms, in the way of subobjects. To be more specific the interpretation of a syntactical entity will be a large diagram $\Psi{:}D \rightarrow \mathbb{C}$, a glorified decorated template, where the graph D can be divided into two parts:

a) An "interesting" part, which is the one we look at (i.e. the chain of display maps of a context), and whose non-basic components (these components being subdiagrams) are defined only up to unique isomorphism.

b) An auxiliary part, which is what guarantees that the components of the interesting part are defined (induction is involved) only up to *unique* isomorphism.

In other words the auxiliary part guarantees that given two diagrams $\Psi,\Psi'{:}D \rightarrow \mathbb{C}$ that are valid interpretations, there exists a unique isomorphism $\Psi \rightarrow \Psi'$. For example, an "interesting" component could be an edge α such that $D\alpha$ is a display map F^*X which is a pullback, while its "auxiliary" counterpart would be the other leg of the pullback. There is some amount of technical work that has to be done to prove that the interpretation of a judgement is invariant from the way this judgement has been constructed: a coherence theorem. These technical matters will await a subsequent paper [La2] . Any category theorist worth his salt will see that coherence is "true and obvious", since all the operations we use are universal constructions, and the Beck condition always holds when quantifiers are involved. From now on, with the exception of 4.12, we will only look at the "interesting" part of an interpretation and call it <u>the</u> interpretation.

Some of the formation rules are common to all display categories and can be called the <u>structural</u> <u>rules</u> for types. In what follows whenever we mention the context Γ it is $x_0{:}X_0, \ldots, x_n{:}X_n$ and has interpretation

$$\mathbf{1} \xleftarrow{X_0} \mathbf{X}_0 \xleftarrow{X_1} \mathbf{X}_1 \cdots \xleftarrow{X_{n-1}} \mathbf{X}_{n-1} \xleftarrow{X_n} \mathbf{X}_n \;,$$

2.7 The structural rules are

(Basic) All basic contexts and judgements are contexts and judgements of $\mathfrak{C}$.

(ChgBs) [WWWPullback rule for terms and types defined in subcontexts]

(Subs) If Γ is a context and $[x_0{:}X_0, \ldots, x_{i-1}{:}X_{i-1}]s{:}X_i$ is a judgement (the types and variables forming an initial segment of Γ) then $\Lambda = x_0{:}X_0, \ldots, x_{i-1}{:}X_{i-1}, x_{i+1}{:}X_{i+1}[x_i \leftarrow s], \ldots x_n{:}X_n[x_i \leftarrow s]$ is a context, where $A[x \leftarrow b]$ means that every occurence of x in A is replaced by b . As usual, one has to insure that no free variable of s will come under the scope of a quantifier; this can be achieved by renaming things. The interpretation of the new contxt is the sequence $Y_0, \ldots, Y_{n-1}$ of display maps, where $Y_j = X_j$ for $j < i$ and

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{X}_i & \xleftarrow{X_{i+1}} & \mathbf{X}_{i+1} & \xleftarrow{X_{i+2}} \cdots & \mathbf{X}_{n-1} & \xleftarrow{X_n} & \mathbf{X}_n \\
 & & & & & & \downarrow\uparrow s & & \uparrow & & \uparrow & & \uparrow q \\
\mathbf{1} & \xleftarrow[Y_0]{X_0} & \mathbf{X}_0 & \xleftarrow[Y_1]{X_1} & \mathbf{X}_1 \cdots & \xleftarrow[Y_{i-1}]{X_{i-1}} & \mathbf{X}_{i-1} & \xleftarrow[Y_i]{} & \mathbf{Y}_i & \xleftarrow[Y_{i+1}]{} \cdots & \mathbf{Y}_{n-2} & \xleftarrow[Y_{n-1}]{} & \mathbf{Y}_{n-1}
\end{array}$$

$Y_i, Y_{i+1}, \ldots, Y_{n-1}$ are obtained by pulling back $X_{i+1}, \ldots, X_n$ by s , as the diagram shows. If $[\Gamma]A$ and $[\Gamma]t{:}A$ are a type and a term judgement where the interpretation of A is $A{:}\mathbf{A} \to \mathbf{X}_n$ and that of t the section $t{:}\mathbf{X}_{n-1} \to \mathbf{X}_n$ of A , then $[\Lambda]t[x_i \leftarrow s]{:}A[x_i \leftarrow s]$ and $[\Lambda]A[x_i \leftarrow s]$ are judgements, with the interpretation of $A[x_i \leftarrow s]$ being the pullback $B{:}\mathbf{B} \to \mathbf{Y}_{n-1}$ $q'{:}\mathbf{B} \to \mathbf{A}$ of A by q , and that of $t[x_i \leftarrow s]$ the unique $t'{:}\mathbf{Y}_{n-1} \to \mathbf{B}$ with $q't' = tq$. If $[\Gamma]_p\varphi$ is a predicate judgement then $[\Lambda]_p\varphi[x_i \leftarrow s]$ is a predicate judgement, the interpretation of $\varphi[x_i \leftarrow s]$ being pullback by q .

(Iden) If Γ is a context then $[\Gamma]x_i{:}X_i$ is a context, the interpretation of the second occurence of X_i being the pullback $X{:}\mathbf{X}\to\mathbf{X}_n$ of X_i by $X_i\circ X_{i+1}\circ\ldots\circ X_n$, with $X'{:}\mathbf{X}\to\mathbf{X}_i$ as the other leg of the pullback, and the interpretation of the second occurence of x_i being the section $\delta{:}\mathbf{X}_n\to\mathbf{X}$ of X such that $X'\circ\delta = X_{i+1}\circ\ldots\circ X_n$.

(NwCn) [Introduction of new atomic terms named after sections of composite types]

Notice that the combination of (Subs) and (Iden) finally allows us to have repetitions of variables in a term or type. The reader may find it a bit hard at first to chase diagrams whose arrows go in the reverse direction from what is traditional But notice that we are consistent in our approach. In particular, type arrows always go left or down, and term arrows always go right or down. We can now interpret a rudimentary form of logic in our display category, rudimentary because the only connective we are allowed to use yet is conjunction:

(Conjf) If $[\Gamma]_p\varphi$ and $[\Gamma]_p\psi$ are judgements of $\mathfrak{C}$, then $[\Gamma]_p\varphi\wedge\psi$ is a judgement, its interpretation $[\![\varphi\wedge\psi]\!]$ being the intersection of the subobjects $[\![\varphi]\!]$ and $[\![\psi]\!]$.

(Truthf) If Γ is a context then $[\Gamma]_p\mathrm{T}$ is a predicate judgement, its interpretation being the full subobject of $\mathbf{X}_n$.

2.8 Definition

Given predicate judgements $[\Gamma]_p\varphi_1,\ldots,[\Gamma]_p\varphi_m$ and $[\Gamma]_p\psi$ we write $[\Gamma]\,\varphi_1,\ldots,\varphi_n\vDash\psi$ to mean that the intersection (infimum) $[\![\varphi_1]\!]\cap\ldots\cap[\![\varphi_n]\!]$ of the interpretations is contained in $[\![\psi]\!]$. We write $[\Gamma]\vDash\psi$ to mean that $[\![\psi]\!]$ is the full subobject.

We have the usual deduction rules:

[(Weak),(Exch),(Cut),(Conj), $[\Gamma]\vDash$ WWW]

2.9 Definition

Let $(\mathfrak{C},\mathfrak{F})$ be a display category. We say that $(\mathfrak{C},\mathfrak{F})$ admits products if for every $F{:}\mathbf{F}\to\mathbf{Y}$ in $\mathfrak{F}$ the pullback functor $F^*{:}\mathfrak{F}_{\mathbf{Y}}\to\mathfrak{F}_{\mathbf{F}}$ has a right adjoint Π_F, and the Beck(-Chevalley) condition holds for pullback diagrams with two parallel display maps: if

$$\begin{array}{ccc} & S & \\ \mathbf{E} & \longrightarrow & \mathbf{F} \\ E\downarrow & & \downarrow F \\ \mathbf{X} & \longrightarrow & \mathbf{Y} \\ & X & \end{array}$$

is a pullback where E, F and $G{:}\mathbf{G}\to\mathbf{F}$ are in $\mathcal{F}$, then the natural morphism $X^*\Pi_F G \to \Pi_E S^*G$ in $\mathcal{F}_{\mathbf{X}}$ is an iso.

This gives us a type formation rule:

(Prodf)
$$\frac{[x_0{:}X_0, \ldots, x_n{:}X_n, x{:}X]\ A}{[x_0{:}X_0, \ldots, x_n{:}X_n]\ \Pi_{x:X} A}$$

along with one term formation rule and one new term:

(λ-abs)
$$\frac{[x_0{:}X_0, \ldots, x_n{:}X_n, x{:}X]\, t{:}A}{[x_0{:}X_0, \ldots, x_n{:}X_n](\lambda x{:}X)\, t : \Pi_{x:X} A}$$

(eval)
$$[x_0{:}X_0, \ldots, x_n{:}X_n, y{:}X, z{:}\Pi_{x:X}A]\, z\cdot y : A \quad .$$

These judgements are interpreted as follows: first denote $x_0{:}X_0, \ldots, x_n{:}X_n$ by Γ and assume that the interpretation of $[\Gamma, x{:}X]\, t{:}A$ is the diagram

$$\begin{array}{ccccccccccc} & & X_0 & & X_1 & \cdots & & X_n & & X & & A \\ \mathbf{1} & \longleftarrow & \mathbf{X}_0 & \longleftarrow & \mathbf{X}_1 & \ldots & \longleftarrow & \mathbf{X}_n & \longleftarrow & \mathbf{X} & \underset{t}{\rightleftarrows} & \mathbf{A} \end{array}$$

Then the interpretation of $[\Gamma](\lambda x{:}X)t : \Pi_{x:X}A$ is the diagram

$$\begin{array}{ccccccccc} & & X_0 & & X_1 & \cdots & & X_n & \Pi_X A \\ \mathbf{1} & \longleftarrow & \mathbf{X}_0 & \longleftarrow & \mathbf{X}_1 & \ldots & \longleftarrow & \mathbf{X}_n & \underset{t'}{\rightleftarrows} \cdot \end{array}$$

where t' is the unique morphism $1_{X_n} \to \Pi_X A$ in $\mathcal{F}_{\mathbf{X}_n}$ such that $ev \circ X^*(t') = t$, $ev{:}X^*(\Pi_X A) \to A$ being the counit of the adjunction. The interpretation of $[\Gamma, y{:}X, z{:}\Pi_{x:X}A]\, z\cdot y : A$ is the diagram

$$\begin{array}{ccccccccccccc} & & X_0 & & X_1 & \cdots & & & X & & Y & & g \\ \mathbf{1} & \longleftarrow & \mathbf{X}_0 & \longleftarrow & \mathbf{X}_1 & \ldots & \mathbf{X}_n & \longleftarrow & \mathbf{X} & \longleftarrow & \cdot & \underset{Y^*A}{\rightleftarrows} & \cdot \end{array}$$

where Y is the pullback $X^*(\Pi_X A)$ and g the section of Y^*A determined by ev, i.e. $qg = ev$

$$\begin{array}{ccc} & q & \\ \mathbf{A} & \longleftarrow & \cdot \\ A\downarrow & \nwarrow ev & \downarrow Y^{*}A \\ \mathbf{X} & \longleftarrow & \cdot \\ & & Y = X^{*}(\Pi_X A) \end{array}$$

Instead of writing $z \cdot y$ we will often use $z(y)$, the more conventional notation. Notice that β- and η-reduction are true in $\mathcal{C}$, because they are just the syntactical translation of adjunction, and so, given $[\Gamma]s{:}X$ we should have the right to state $[\Gamma]((\lambda x{:}X)t)\cdot s = t[x \leftarrow s]$ and $[\Gamma, z{:}\Pi_{x:X}A]((\lambda x{:}X)z\cdot x) = z$. But we have no way of knowing if the equality symbols $=_A$ and $=_{(\ldots)}$ are available in our model. The author did some amount of soul searching before reaching the conclusion that there *should be another notion of equality, not a predicate, for expressing the equality of parallel maps of the model.* Let us use the symbol $\equiv$ for that new equality. The author then learned, much to his surprise, that Martin-Löf, whose concerns were completely different, had reached the same conclusion and had defined an "external", as opposed to an "internal" notion of equality. We will not formalize the use of the external $\equiv$ here, since we do not need it to discuss the concepts we want to deal with; we will only give an informal discussion of the whys and hows. First, a notion like $\equiv$ is obviously needed if one wants to complete the Lawvere program for dependent type theory and construct categories from formal systems. At one point or others terms will have to be identified by a congruence to give morphisms, and so for every type X there will be a $\equiv_X$. But the use of $\equiv$ will be strictly regulated; for example if x, y are distinct variables the statement $x \equiv y$ will be proscribed. This would mean that (using our favorite metaphor) given a class of mathematical structures, we could pick two structures in it at random and decide if they are equal or not. But mathematical structures have automorphisms, so they can "rotate", and we have no way of knowing how to "orient" two structures to compare them. On the other hand the statement $x \equiv x$ will be legal, because once we get hold a structure, we can hold it firmly!

2.10 Proposition

Let $(\mathcal{C}, \mathcal{F})$ be a display category that admits products and $\mathcal{K} \subset \mathcal{F}$ a class of maps which is closed under composition, pullbacks by arbitrary maps, and the right adjoint of pullback by arbitrary maps.

Then for every $\mathbf{X} \in \mathcal{C}$ the full subcategory $\mathcal{K}_{\mathbf{X}} \subset \mathcal{F}_{\mathbf{X}}$ whose objects are the maps in $\mathcal{K}$ is cartesian closed and for every *map* $F{:}\mathbf{Y} \to \mathbf{X}$ the pullback functor F^* preserves the cartesian closed structure.

Let $A{:}\mathbf{A} \to \mathbf{X}$, $B{:}\mathbf{B} \to \mathbf{X}$ and $C{:}\mathbf{C} \to \mathbf{X}$ be in $\mathcal{K}_{\mathbf{X}}$. The product $A \times B$ in $\mathcal{K}_{\mathbf{X}}$ is the diagonal P of the pullback

$$\begin{array}{ccc} \mathbf{P} & \longrightarrow & \mathbf{B} \\ \downarrow & & \downarrow \\ \mathbf{A} & \longrightarrow & \mathbf{X} \end{array} \quad ,$$

which is in $\mathcal{K}$ by assumption. Let $B \Rightarrow C = \prod_B B^*C$. We get

$$\begin{aligned} \mathcal{K}_{\mathbf{X}}(A \times B, C) &= \mathcal{K}_{\mathbf{X}}(P, C) \cong \mathcal{K}_{\mathbf{X}}(B \circ B^*A, C) \cong \mathcal{K}_{\mathbf{B}}(B^*A, B^*C) \\ &\cong \mathcal{K}_{\mathbf{B}}(A, \textstyle\prod_B B^*C) \end{aligned}$$

and this shows $\mathcal{K}_{\mathbf{X}}$ is cartesian closed. Let $F{:}\mathbf{Y} \to \mathbf{X}$. It is trivial to show $F^*{:}\mathcal{K}_{\mathbf{X}} \to \mathcal{K}_{\mathbf{Y}}$ preserves products. To show F^* preserves exponentiation use the Beck condition on the pullback square

$$\begin{array}{ccc} \mathbf{P} & \longrightarrow & \mathbf{B} \\ \downarrow & & \downarrow B \\ \mathbf{Y} & \underset{F}{\longrightarrow} & \mathbf{X} \end{array} \quad . \qquad \square$$

The construction of $B \Rightarrow C$ above inspires the following notation: if B, C are two types, and if x is a variable that does not appear in C, then the type $B \Rightarrow C$ will stand for $\prod_{x:B} C$.

The following is very useful technically:

2.11 Proposition (Streicher [St])

Let $(\mathcal{C}, \mathcal{F})$ be a display category. Then the following are equivalent:

i) For every display map $F{:}\mathbf{F} \to \mathbf{Y}$ the pullback functor $F^*{:}\mathcal{C}/\mathbf{Y} \to \mathcal{C}/\mathbf{F}$ has a partial left adjoint $\prod_F$ which is defined for every object of $\mathcal{F}_{\mathbf{F}} \subset \mathcal{C}/\mathbf{F}$ and lands in $\mathcal{F}_{\mathbf{Y}}$. In other words, for every display map $G{:}\mathbf{G} \to \mathbf{F}$ there is a display map $\prod_F G{:}\prod_F \mathbf{G} \to \mathbf{Y}$ and $\varepsilon G{:}F^*\prod_F G \to G$ in $\mathcal{F}_{\mathbf{F}}$ such that εG is (co)universal: for every $H{:}\mathbf{H} \to \mathbf{Y}$ in $\mathcal{C}$ we have the isomorphism $\mathcal{C}/\mathbf{Y}(H, \prod_F G) \cong \mathcal{C}/\mathbf{F}(F^*H, G)$, mediated in the usual way.

ii) $\mathcal{F}$ admits products.

For i)⇒ii) we obviously only have to prove the Beck condition. Let E,F,S,X be the same pullback square as above, and let $G{:}\mathbf{G}\to\mathbf{F}$ and $A{:}\mathbf{A}\to\mathbf{X}$ be display maps. Let $E^*A{:}\mathbf{B}\to\mathbf{E}$ be the pullback.

$$\begin{array}{ccccccc} \mathbf{B} & \xrightarrow{E^*A} & \mathbf{E} & \xrightarrow{S} & \mathbf{F} & \xleftarrow{G} & \mathbf{G} \\ {\scriptstyle A^*E}\downarrow & & {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle F} & & \\ \mathbf{A} & \xrightarrow[A]{} & \mathbf{X} & \xrightarrow[X]{} & \mathbf{Y} & & \end{array}$$

We get

$$\begin{array}{lll} \mathcal{F}_{\mathbf{X}}(A,X^*\Pi_F G) & \cong \mathcal{C}/\mathbf{Y}(XA,\Pi_F G) & \text{by pullback} \\ & \cong \mathcal{C}/\mathbf{F}(S\circ E^*A,G) & \text{by assumption} \\ & \cong \mathcal{F}_{\mathbf{E}}(E^*A,S^*G) & \text{By pullback} \\ & \cong \mathcal{F}_{\mathbf{X}}(A,\Pi_E S^*G) & , \end{array}$$

and this being true for any A, proves the claim. For the converse, let $F{:}\mathbf{F}\to\mathbf{Y}$, $G{:}\mathbf{G}\to\mathbf{F}$ be display maps. We have to show that for any $X{:}\mathbf{X}\to\mathbf{Y}$ in $\mathcal{C}$ $\mathcal{C}/\mathbf{Y}(X,\Pi_F G)\cong\mathcal{C}/\mathbf{F}(F^*X,G)$. Let $S{:}\mathbf{E}\to\mathbf{F}$, $E{:}\mathbf{E}\to\mathbf{X}$ be the pullback of F and X. Then

$$\begin{array}{lll} \mathcal{C}/\mathbf{Y}(X,\Pi_F G) & \cong \mathcal{C}/\mathbf{X}(1_X,X^*\Pi_F G) & \text{by pullback} \\ & = \mathcal{F}_{\mathbf{X}}(1_X,X^*\Pi_F G) & \text{by Unit} \\ & \cong \mathcal{F}_{\mathbf{X}}(1_X,\Pi_E S^*G) & \text{by Beck} \\ & \cong \mathcal{F}_{\mathbf{Y}}(1_E,S^*G) & \text{by } \Pi_E\text{-adjunction} \\ & \cong \mathcal{C}/\mathbf{F}(S,G) & \text{by pullback} \\ & = \mathcal{C}/\mathbf{F}(F^*X,G) \quad . & \text{QED. } \square \end{array}$$

2.12 Definition

Let $(\mathcal{C},\mathcal{F})$ be a display category. We say that it <u>admits</u> <u>sums</u> if $\mathcal{F}$ is closed under composition.

Note that it follows trivially from this that for every $F{:}\mathbf{F}\to\mathbf{Y}$ in $\mathcal{F}$ the pullback functor $F^*{:}\mathcal{F}_{\mathbf{Y}}\to\mathcal{F}_{\mathbf{F}}$ has a left adjoint $\Sigma_F = F\circ(-)$, and the Beck condition holds for pullback diagrams with two parallel display maps. But the converse is not true in general. If $(\mathcal{C},\mathcal{F})$ admits sums we have access to the following formation rules and terms (using the same notation for Γ,X,t as in 2.9):

(Sumf) $$\frac{[\Gamma, x{:}X]\,A}{[\Gamma]\,\Sigma_{x:X} A}$$

(Pair) $$\frac{[\Gamma] s{:}X \quad [\Gamma]\, t{:}A[x \leftarrow s]}{[\Gamma] \langle s,t \rangle{:}\,\Sigma_{x:X} A}$$

(Proj1) $$[\Gamma, y{:}\Sigma_{x:X} A]\, \pi(y){:}X$$

(Proj2) $$[\Gamma, y{:}\Sigma_{x:X} A]\, \pi'(y){:}A(\pi(y))$$

Supposing that the interpretation of s is the section s of X, that of t the section t of s^*A then the interpretation of $\langle s,t \rangle{:}\Sigma_{x:X} A$ is the map qt, where q is the other leg of the pullback

$$\begin{array}{ccc} & A & \\ \mathbf{X} & \longleftarrow & \mathbf{A} \\ X \downarrow\uparrow s & t & \uparrow q \\ \mathbf{X_n} & \rightleftarrows & \mathbf{A} \\ & s^*A & \end{array} .$$

The interpretation of $\pi(y){:}X$ is the section π of $(XA)^*X$ determined by the morphism $A{:}XA \to X$ of $\mathcal{F}/\mathbf{X_n}$.

$$\begin{array}{ccc} \mathbf{X} & \longleftarrow & \cdot \\ X \downarrow & \nwarrow A & \downarrow XA^*X \\ \mathbf{X_n} & \longleftarrow & \mathbf{A} \\ & & XA = \Sigma_{x:X} A \end{array} .$$

By (Subs) the interpretation of $A(\pi(y))$ is the pullback A^*A. The interpretation of $\pi'(y)$ is then the diagonal of A^*A

$$\begin{array}{ccccc} & & A & & \\ & & \mathbf{X} & \longleftarrow & \cdot \\ & X \swarrow & \uparrow A & & \uparrow \\ \mathbf{X_n} & \longleftarrow & \mathbf{A} & \longleftarrow & \cdot \\ & \Sigma_{x:X} A = XA & & A^*A & \end{array} .$$

The reader can verify the equations $[\Gamma]\, \pi(\langle s,t \rangle) \equiv_X s$, $[\Gamma]\, \pi'(\langle s,t \rangle) \equiv_{A(\pi(s))} t$ and $[\Gamma, y{:}\Sigma_{x:X} A]\, \langle \pi(y), \pi'(y) \rangle \equiv_{(\dots)} y$, which we give for the record.

One consequence of the definitions is that if x does not appear in A we can denote $\Sigma_{x:X}A$ by $X \times A$. Then the pair $\langle s,t \rangle$ is the ordinary cartesian pair.

2.13

If a display category admits sums there is another logical connective at our disposal: Let $[\Gamma, x{:}X]_p\, \varphi$ be a predicate judgement such that

$$[\Gamma, x{:}X, y{:}Y]\, \varphi, \varphi[x \leftarrow y] \models x = y \qquad (*) \quad .$$

Then we can introduce the predicate $\exists!_{x:X}\varphi$, with the judgement $[\Gamma]_p \exists!_{x:X}\varphi$. The point is that if $X{:}\mathbf{X} \to \mathbf{X}_n$ is the interpretation of X and $[\![\varphi]\!] \hookrightarrow \mathbf{X}$ is that of φ, the sequent $(*)$ asserts that the composite $[\![\varphi]\!] \hookrightarrow \mathbf{X}_n$ is a monomorphism, and since it is a display map it can be used as the interpretation of $\exists!_{x:X}\varphi$. The deduction rules for $\exists!$ are those of $\exists$ (4.2). This allows us, among other things to give a *comprehension scheme* that turns predicates into types. Let $[\Gamma]_p \psi$ be a predicate judgement. Since its interpretation is a display map, it also corresponds to a type: $[\Gamma]\, T_\psi$. This type always has an equality symbol (the diagonal of a mono is the identity) and so we can write $[\Gamma, x{:}T_\psi, y{:}T_\psi]\, x = y$, which internally asserts that the display map is a mono; all this can be done in any display category, but the following cannot: we get

$$[\Gamma]\, \exists!_{x:T_\psi} \mathrm{T} \models \psi \qquad \text{and} \qquad [\Gamma]\, \psi \models \exists!_{x:T_\psi} \mathrm{T} \quad ,$$

which internally asserts the validity of the comprehension scheme.

Another thing that $\exists!$ allows us to do is give a notion of term descriptor. Let $[\Gamma, x{:}X]_p\, \varphi$ as above be such that $[\Gamma] \models \exists!_{x:X}\varphi$. This asserts that the composite $[\![\varphi]\!] \hookrightarrow \mathbf{X} \to \mathbf{X}_n$ is an isomorphism. This isomorphism enables us to construct a section $t{:}\mathbf{X}_n \to \mathbf{X}$ of X and so if we want we can add a term formation rule

(Descr) $$\frac{[\Gamma] \models \exists!_{x:X}\varphi}{[\Gamma]\, (\delta x)\varphi : X}$$

where δ is a binding operator meaning "the unique x such that φ", and thus

$$[\Gamma] \models \varphi((\delta x)\varphi) \quad .$$

§ 3 THE "ESSENTIALLY ALGEBRAIC" THEORY

3.1 Definition

A groupoid (or category) is be said to be free if it isomorphic to the free groupoid (or category) generated by a graph. A groupoid (or category) is said to be finitely presented if it can be described with a finite number of objects, generating morphisms and equations between parallel morphisms. If $\mathfrak{E}$ is a category with a sufficient amount of structure (the reader can read this as "an elementary topos" without much loss of generality), we will often call its objects $\mathfrak{E}$-sets and its morphisms $\mathfrak{E}$-functions , in order to emphasize that there exists a semantics that allows us to legitimately reason informally as if the objects were sets, if we make sure we tayle logic we use to the amount of categorical structure available.

Let $\mathcal{G}$ be a groupoid-enriched category, that is, a 2-category all whose 2-cells are isomorphisms. We will sometimes call its objects $\mathcal{G}$-groupoids , the 1-cells between them $\mathcal{G}$-functors , and the 2-cells $\mathcal{G}$-transformations . Unless otherwise noted, whenever we talk about limits in a 2-category, like products or pullbacks, we mean 2-limits: if $(\mathbf{X}_d)_{d\in\mathbb{D}}$ is a diagram in $\mathcal{G}$ seen as an ordinary category, then $\mathbf{Y}$ along with a cone to $(\mathbf{X}_d)_d$ is a 2-limit for $(\mathbf{X}_d)_d$ if given any $\mathbf{A}\in\mathcal{G}$ the usual natural isomorphism

$$\mathcal{G}(\mathbf{A},\mathbf{Y}) \cong \mathit{Cone}(\mathbf{A},(\mathbf{X}_d)_d)$$

is an isomorphism of *groupoids*, not only of set; it is easy to see that the right side of the isomorphism has a natural groupoid structure. For instance, if we say $\mathcal{G}$ has products, we mean that for any $\mathcal{G}$-groupoids $\mathbf{A},\mathbf{X},\mathbf{Y}$, we always have $\mathcal{G}(\mathbf{A},\mathbf{X}\times\mathbf{Y}) \cong \mathcal{G}(\mathbf{A},\mathbf{X})\times\mathcal{G}(\mathbf{A},\mathbf{Y})$ as groupoids. Let I be a (set-)groupoid. Write $\mathcal{G}(\mathbf{X},\mathbf{Y})^{\mathrm{I}}$ for the groupoid of all functors and natural transformations $\mathrm{I}\to\mathcal{G}(\mathbf{X},\mathbf{Y})$. This means that if $\Psi\in\mathcal{G}(\mathbf{X},\mathbf{Y})^{\mathrm{I}}$ and $\sigma:i\to j$ in I there is $\Psi_\sigma:\Psi_i\to\Psi_j$ in $\mathcal{G}(\mathbf{X},\mathbf{Y})$, etc.. Let $F:\mathbf{D}\to\mathbf{X}$ in $\mathcal{G}$. Denote by $\Psi * F$ the object of $\mathcal{G}(\mathbf{D},\mathbf{Y})^{\mathrm{I}}$ that sends $i\in\mathrm{I}$ to $\Psi_i F$. If $\alpha:F\to F'$ in $\mathcal{G}(\mathbf{D},\mathbf{X})$ there is an obvious $\Psi*\alpha:\Psi*F\to\Psi*F'$. We leave the following for the reader to prove: If $\mathbf{C}\in\mathcal{G}$ and $\beta:G\to G'$ is in $\mathcal{G}(\mathbf{C},\mathbf{D})$ then

$$\Psi*(FG) = (\Psi*F)*G \quad , \quad \Psi*(F\beta) = (\Psi*F)*\beta \qquad .$$

We say $\mathcal{G}$ has finite cotensors [Ke] if, given a finitely presented groupoid I , and a $\mathcal{G}$-groupoid $\mathbf{X}$, there exists $\mathbf{X}^I \in \mathcal{G}$ along with $\Phi: I \to \mathcal{G}(\mathbf{X}^I, \mathbf{X})$ with the following universal property: for any $\mathbf{A} \in \mathcal{G}$, the morphism of groupoids $\mathcal{G}(\mathbf{A}, \mathbf{X}^I) \to \mathcal{G}(\mathbf{A}, \mathbf{X})^I$ which sends F to $\Phi * F$ is an isomorphism. The paradigm for cotensors is the arrow (comma) object: suppose $\mathcal{G}$ has finite cotensors; if **2** is the free groupoid with two objects 0,1 and one (iso)morphism $0 \to 1$, then for any $\mathbf{X} \in \mathcal{G}$ $\mathbf{X}^2$ comes equipped with $d_0, d_1: \mathbf{X}^2 \to \mathbf{X}$, and $\rho: d_0 \to d_1$, such that for any $\mathbf{A} \in \mathcal{G}$, $\alpha: X \to Y$ in $\mathcal{G}(\mathbf{A}, \mathbf{X})$, there exists a unique $a: \mathbf{A} \to \mathbf{X}^2$ with $\rho a = \alpha$. Let $K: I \to J$ be a morphism of finitely presented groupoids. For $\mathbf{X} \in \mathcal{G}$ there is $\mathbf{X}^K: \mathbf{X}^J \to \mathbf{X}^I$ which is defined by the equation $\Phi * \mathbf{X}^K = \Psi K$, where Φ, Ψ are the universal diagrams of $\mathbf{X}^I, \mathbf{X}^J$ respectively. From now on $\mathcal{G}$ will be assumed to have finite products and cotensors.

Let $\mathbf{C} \in \mathcal{G}$. A fibration above $\mathbf{C}$ is a $\mathcal{G}$-functor $E: \mathbf{E} \to \mathbf{C}$ such that, given any $\mathbf{A} \in \mathcal{G}$, $X', X: \mathbf{A} \to \mathbf{E}$ and $\alpha: X' \to X$, then for any $Y: \mathbf{A} \to \mathbf{E}$ such that $EY = X$ (" Y is above X ", or " Y is in the fiber of X") there exist $Y': \mathbf{A} \to \mathbf{E}$, $\beta: Y' \to Y$ such that $EY' = X'$ and $E\beta = \alpha$. A fibration E is discrete if any $\mathbf{A}, X, X', Y, \alpha$ as above determine a unique Y', β with the required property. A simple translation argument shows that this is the same as saying that whenever $\gamma: Z' \to Z$ is such that $EZ' = EZ$ and $E\gamma$ is the identity then already $Z' = Z$ and $\gamma = 1_Z$ ("every fiber is discrete".) Given another fibration $F: \mathbf{F} \to \mathbf{C}$, a morphism of fibrations $H: E \to F$ is a $\mathcal{G}$-functor $H: \mathbf{E} \to \mathbf{F}$ such that $FH = E$.

As has been repeatedly said, the intuition behind our approach is that an object $\mathbf{C}$ of $\mathcal{G}$ should be thought of as a class of mathematical structures, which are the "elements", in the Kripke-Joyal sense, of $\mathbf{C}$, along with the isomorphisms between these elements/structures, the whole forming a groupoid. The fibrations will be the display maps: a fibration $F: \mathbf{F} \to \mathbf{C}$ is an indexed family $(X_c)_{c \in \mathbf{C}}$ of groupoids X_c , indexed by the groupoid $\mathbf{C}$. If $c \to d$ is a morphism in $\mathbf{C}$ it is quite natural to require that there be an *equivalence* $X_c \to X_d$, since groupoids have more structure that sets and we have to take that structure into account. If F above is discrete, this means that all the X_c are discrete, i.e. sets (or classes). A discrete fibration of groupoids has another intuitive interpretation, due to Joyal. Think of the groupoid $\mathbf{F}$ as a class of structures which are richer than those of $\mathbf{C}$, and of F as a forgetful functor, say from rings to abelian groups. Then a set X_c is the class of all $\mathbf{F}$-structures definable on the underlying structure c , and the action of $\mathbf{C}$ corresponds to *transport of structure* along an isomorphism [Jo] . The fact that F is discrete means that the structure $\mathbf{C}$ acts as a "sufficient support" for the structure $\mathbf{F}$. An arbitrary fibration F can also be considered as a forgetful functor, of a more general sort; for example one sees that the forgetful functor from the groupoid of small categories and isomorphisms to the groupoid of sets and bijections that sends a small category to its set of objects is a fibration, but not a discrete one; this means that some important information is lost by not looking at the morphisms of the small categories.

$\mathcal{G}$ can be endowed with intuitive contents in another way. Given a groupoid $\mathbf{G}$, one can think of it as a given class of concepts to be realized by terms, which are the objects of $\mathbf{G}$. A morphism of $\mathbf{G}$ is to be thought of as a proof that two terms realize the same concept. A fibration $\mathbf{G}\to\mathbf{H}$ is a mapping between terms and proofs such that proofs in $\mathbf{H}$ can be lifted to $\mathbf{G}$. We feel this approach should have applications in proof theory (G. Huet also proposed this idea [Hu] .)

3.2 Proposition

Let $E:\mathbf{E}\to\mathbf{C}$ be a fibration in $\mathcal{G}$, $G:\mathbf{D}\to\mathbf{C}$ a $\mathcal{G}$-functor.

a) If the (2-)pullback $P:\mathbf{P}\to\mathbf{D}$ exists, it is a fibration. If E is discrete, P is discrete too.

b) Any projection $\mathbf{A}\times\mathbf{C}\to\mathbf{C}$ is a fibration. In particular the unique $\mathbf{A}\to\mathbf{1}$ to the terminal object is a fibration.

c) The composite of two fibrations is a fibration, and the composite of two discrete fibrations a discrete fibration.

d) If $F:\mathbf{F}\to\mathbf{C}$ is a discrete fibration, then any morphism $H:E\to F$ in $\mathcal{F}_{\mathbf{C}}$ is a fibration $\mathbf{E}\to\mathbf{F}$; if E is also discrete, H is discrete.

e) Any monomorphism $\mathbf{X}\to\mathbf{C}$ which is a fibration is a discrete one.

f) If $\mathcal{G}$ has finite cotensors and I is a f.p. groupoid then $E^I:\mathbf{E}^I\to\mathbf{C}^I$ is a fibration. If E is discrete E^I is discrete.

We will do the first part of d) and leave the others, all trivial, to the reader. Let $X, X' : \mathbf{B} \to \mathbf{F}$, $\alpha : X' \to X$ and $Y : \mathbf{B} \to \mathbf{E}$ with $HY = X$. Since E is a fibration there is $Y' : \mathbf{B} \to \mathbf{E}$ and $\beta : Y' \to Y$ such that $E\alpha = F\beta$, in particular $EY' = FX'$. Then both α and $H\beta$ are sent by F to $F\beta$ and so $H\beta = \alpha$ since F is discrete.

A $\mathcal{G}$-groupoid $\mathbf{A}$ is said to be discrete if $\mathbf{A} \to \mathbf{1}$ is a discrete fibration, i.e. if $\mathcal{G}(\mathbf{X}, \mathbf{A})$ is a discrete groupoid for any X. Obviously any morphism between discrete objects is a discrete fibration.

The presence of discrete fibrations will add one feature to the type theory, in that we will distinguish between ordinary fibrations and discrete ones, since discrete types are obviously very important. It bears repeating that a fibration which is a monomorphism is always discrete; we call such monos replete monos; we know they will be used to interpret predicates.

3.3 Examples

Take $\mathcal{G}$ to be the category of small groupoids. Then a fibration $E : \mathbf{E} \to \mathbf{C}$ in $\mathcal{G}$, as defined, is just a Grothendieck fibration of groupoids. That is, if $x' \to x$ is morphism of $\mathbf{C}$ and $y \in \mathbf{E}$ such that $Ey = x$ then there exists y', $\beta : y' \to y$ such that $E\beta = \alpha$. To derive this from the abstract definition, just put $\mathbf{A} = \mathbf{1}$. The converse is easy to prove too, but notice that it requires the axiom of choice. A discrete fibration in $\mathcal{G}$ is just a Grothendieck fibration $E : \mathbf{E} \to \mathbf{C}$ where for every $y \in \mathbf{C}$ the fiber $\mathbf{E}^c = E^{-1}(c)$ is discrete.

Let $\mathbb{C}$ be a small category. We recall that $\mathrm{Fib}_G/\mathbb{C}$ is the 2-category whose objects are diagrams $\mathbf{X} : \mathbb{X} \to \mathbb{C}$ of small categories that are fibrations of groupoids. Our convention on notation should be clear: we use a bold letter for an object of the category, i.e. a fibration above $\mathbb{C}$, and the corresponding "blackboard bold" letter for the total category of the fibration. A morphism $F : \mathbf{X} \to \mathbf{Y}$ is a functor $F : \mathbb{X} \to \mathbb{Y}$ between the underlying categories such that $\mathbf{Y}F = \mathbf{X}$. Obviously, it is always a cartesian functor. Given $F, G : \mathbf{X} \to \mathbf{Y}$, a 2-cell $F \to G$ is a natural transformation $\alpha : F \to G$ above identity, i.e. such that $\mathbf{Y}\alpha = 1_{\mathbf{X}}$. This makes $\mathrm{Fib}_G/\mathbb{C}$ into a groupoid-enriched category, since $\mathbf{Y}$ reflects isos. Let us show that in $\mathrm{Fib}_G/\mathbb{C}$, $F : \mathbf{X} \to \mathbf{Y}$ is a fibration iff the corresponding functor $\mathbb{X} \to \mathbb{Y}$ is itself a Grothendieck fibration (which makes it automatically a fibration of groupoids).
First we claim

3.4 Proposition

Let $F:\mathbf{X}\to\mathbf{Y}$ be as above. Then every morphism of $\mathbb{X}$ is cartesian for F . □

The proof is left to the reader. Now, assume that F is a fibration in $Fib_G/\mathbb{C}$. Let $X\in\mathbb{X}$, and $a:A\to FX$ in $\mathbb{Y}$. Assume that X is above S in $\mathbb{C}$, and that a is above $s:T\to S$. For every $r:R\to S$ in $\mathbb{C}$ choose $\gamma_r:r^*X\to X$ above r in $\mathbb{X}$. This defines a functor $K_\gamma:\mathbb{C}/S\to\mathbb{X}$, which is a morphism of fibrations $U_S\to\mathbf{X}$, where $U_S:\mathbb{C}/S\to\mathbb{C}$ is the standard discrete fibration that sends $r:R\to S$ to R . In other words, U_S is the representable contravariant functor determined by S ; there is a Yoneda lemma for fibrations which asserts that the fiber groupoid $\mathbb{X}^S$ is equivalent to the groupoid of morphisms of fibrations $U_S\to\mathbf{X}$, and this equivalence is mediated by choices γ of cartesian arrows. Look at $FK_\gamma:\mathbb{C}/S\to\mathbb{Y}$. $F\gamma_s$ is above s , and so is a . Therefore there is a unique $\alpha:Fs^*X\to A$ above 1_T such that $a\alpha=\gamma_s$. Define $H:\mathbb{C}/S\to\mathbb{Y}$ along with $\bar{\alpha}:K_\gamma\to H$, as follows:

$$Hr=FK_\gamma r=F(r^*X) \text{ and } \bar{\alpha}r \text{ is identity if } r\neq s$$
$$Hs=A \text{ and } \bar{\alpha}s \text{ is } \alpha .$$

H is obviously a morphism of fibrations and $\bar{\alpha}$ a transformation above identity. By assumption, there is $L:\mathbb{C}/S\to\mathbb{X}$ and $\beta:K_\gamma\to L$, with $F\beta=\alpha$. Then βs is an iso $s^*X\to A$, hence $\gamma_s(\beta s)^{-1}$ is a morphism above a , which proves that F as a functor is a fibration. The converse, which asserts that if F is a Grothendieck fibration then it is a fibration in $\mathcal{G}(\mathbb{C})$, is left to the reader. Here again the axiom of choice is used.

Let us now show that F is a discrete fibration in $Fib_G/\mathbb{C}$ iff F is a discrete fibration as a functor. First assume F is a discrete fibration in the traditional sense. Let

$$\mathbf{Y}\xleftarrow{H}\mathbf{A}\xrightarrow{K}\mathbf{X}$$

in $Fib_G/\mathbb{C}$, along with $\alpha:FK\to H$. By the above we know that there is $L:\mathbf{A}\to\mathbf{Y}$ and $\beta:H\to L$ with $F\beta=\alpha$. Let $L':\mathbf{A}\to\mathbf{Y}$ and $\beta':H\to L'$ also give $F\beta'=\alpha$. For every $A\in\mathbb{A}$, βA and $\beta'A$ are isomorphisms with $F\beta A=\alpha=F\beta'A$. Since F is discrete, we get $\beta A=\beta'A$, and this shows $\beta=\beta'$. For the converse, assume F is a discrete fibration in

$Fib_G/\mathbb{C}$. Let $X \in \mathbb{X}$ be above $S \in \mathbb{C}$ and $a: A \to FX$ above $s: T \to S$. We already know there is $y: Y \to X$ with $Fy = a$. Let $z: Z \to X$ be another morphism with $Fz = a$. By Proposition 1 there is $\alpha: Y \to Z$ with $z\alpha = y$ and $F\alpha = 1_A$. For every $r: R \to S$ in $\mathbb{C}/S$ with $r \neq s$ let $\gamma_r: r^*X \to X$ be a choice of a (cartesian) arrow above r . Define $K, L: \mathbb{C}/S \to \mathbb{X}$ by

$$K(r) = L(r) = r^*X \quad \text{if} \quad r \neq s$$
$$K(s) = Y \ , \ L(s) = Z \quad .$$

The value on morphisms is determined by the cartesian arrows. There is an obvious natural transformation $\bar{\alpha}: K \to L$, which is α on s and identity elsewhere. Obviously, we have $F\bar{\alpha} = 1_{FK} = 1_{FL}$. Since K, L and $\bar{\alpha}$ can be seen to live in $Fib_G/\mathbb{C}$, and F is a discrete fibration therein, we get that $\bar{\alpha}$ is identity, and so $y = z$.

3.5 Proposition

Let $\mathcal{G}$ be a groupoid-enriched category with finite cotensors. Let $K: I \to J$ be a morphism of finitely presented groupoids which is injective on objects. For any $\mathbf{X} \in \mathcal{G}$ the natural $\mathbf{X}^K: \mathbf{X}^J \to \mathbf{X}^I$ is a fibration. If K is bijective on objects, $\mathbf{X}^K$ is a discrete fibration.

Let $\Psi: J \to \mathcal{G}(\mathbf{X}^J, \mathbf{X})$ and $\Phi: I \to \mathcal{G}(\mathbf{X}^I, \mathbf{X})$ be the universal diagrams. Then $\Phi * \mathbf{X}^K = \Psi K$. Let $\alpha: \mathbf{X}^K X \to Y$ be some $\mathcal{G}$-transformation, for $X: \mathbf{A} \to \mathbf{X}^J$ and $Y: \mathbf{A} \to \mathbf{X}^I$. For $j \in J$, define $\Theta_j: \mathbf{A} \to \mathbf{X}$ along with $\vartheta_j: \Psi_j X \to \Theta_j$, as follows:

- If $j = K(i)$ then $\Theta_j = \Phi_i Y$ and $\vartheta_j = \Phi_i \alpha$, i.e. $\vartheta_j: \Psi_j X = \Phi_i \mathbf{X}^K X \longrightarrow \Phi_i Y$.
- If j is not in the image of K , then $\Theta_j = \Psi_j X$ and ϑ_j is the identity.

Now the family $(\Theta_j)_{j \in J}$ can be extended to a diagram $\Theta: J \to \mathcal{G}(\mathbf{A}, \mathbf{X})$. Given $\sigma: j \to k$ in J , one just has to define Θ_σ to be $\vartheta_k \circ \Psi_\sigma X \circ \vartheta_j^{-1}$:

$$\begin{array}{ccc} \Psi_j X & \xrightarrow{\vartheta_j} & \Theta_j \\ {\scriptstyle \Psi_\sigma X} \downarrow & & \downarrow {\scriptstyle \Theta_\sigma} \\ \Psi_k X & \xrightarrow[\vartheta_k]{} & \Theta_k \end{array} \quad .$$

This obviously gives a natural transformation $\theta:\Psi*X\to\Theta$ in $\mathcal{G}(\mathbf{A},\mathbf{X})^J$, and by universality there is $Z:\mathbf{A}\to\mathbf{X}^J$ with $\Psi*Z=\Theta$ and $\beta:X\to Z$ with $\Psi*\beta=\theta$. We then have

$$\Phi*(X^K\beta)=(\Phi*X^K)*\beta=(\Psi K)*\beta \quad ,$$

and so for $\nu:i\to i'$ in I

$$(\Phi*(X^K\beta))_\nu=((\Psi K)*\beta)_\nu=\Psi_{K(\nu)}\beta=\theta_{K(\nu)}=\Phi_\nu\alpha=(\Phi*\alpha)_\nu \quad ,$$

hence $\Phi*X^K\beta=\Phi*\alpha$, and by the universal property of Φ we get $X^K\beta=\alpha$.

Suppose now that K is bijective on objects, and let $X,Z,Z':\mathbf{A}\to\mathbf{X}^J$ along with $\beta:X\to Z$ and $\beta':X\to Z'$ such that $X^K\beta=X^K\beta'=\alpha$. By the universal property of Ψ, it suffices to show $\Psi_\sigma\beta=\Psi_\sigma\beta'$ for any $\sigma:j\to k$ in J. Since K is bijective on objects, by assumption we already have that $\Psi_k\beta=\Psi_k\beta'$. But then

$$\Psi_\sigma\beta=\Psi_k\beta\circ\Psi_\sigma X=\Psi_k\beta'\circ\Psi_\sigma X=\Psi_\sigma\beta' \quad \square$$

3.6 Definition

A <u>trope</u> $\mathcal{G}$ is a groupoid-enriched category with a terminal object, finite cotensors, and such that the (2-)pullback of any fibration by any $\mathcal{G}$-functor exists.

An alternate definition of a trope would be to say that it is a groupoid-enriched category $\mathcal{G}$ with finite cotensors such that $(\mathcal{G},\mathcal{F})$ is a display category, where $\mathcal{F}$ is the class of fibrations. Recall that 3.2 asserts that the class $\mathcal{D}\subset\mathcal{F}$ of discrete fibrations is closed under pullbacks, and that both $\mathcal{D}$ and $\mathcal{F}$ are closed under composition. Given a trope $\mathcal{G}$, for every $\mathbf{X}\in\mathcal{G}$ there is a natural enrichment of $\mathcal{F}_\mathbf{X}$ over groupoids; that is, given $A:\mathbf{A}\to\mathbf{X}$, $B:\mathbf{B}\to\mathbf{X}$ objects of $\mathcal{F}_\mathbf{X}$ and $F,G:A\to B$, a 2-cell $F\to G$ is a $\mathcal{G}$-transformation $\alpha:F\to G$ such that $B\alpha=1_A$. So when we write $\mathcal{F}_\mathbf{X}$ we will mean that we take this 2-categorical structure into account, unless we say otherwise. Notice now that given $H:\mathbf{Y}\to\mathbf{X}$, the adjunction between $H\circ(-)$ and H^* is a 2-adjunction, that is, the natural isomorphism $\mathcal{F}_\mathbf{X}(H\circ(-),-)\cong\mathcal{F}_\mathbf{Y}(-,H^*(-))$ is an isomorphism of groupoids.

We will denote by $\mathcal{D}_\mathbf{X}$ the full subcategory of $\mathcal{F}_\mathbf{X}$ whose obejects are the fibrations. The 2-categorical structure $\mathcal{D}_\mathbf{X}$ inherits from its inclusion in $\mathcal{F}_\mathbf{X}$ is trivial, since 2-cells can only be identity, and so $\mathcal{D}_{\mathbf{X}}$ is "just a category". It is trivial to show $\mathcal{D}_{\mathbf{X}}$ has all finite limits for any $\mathbf{X}$. We will use the notation $\mathcal{R}_{\mathbf{X}}$ for the inf-semilattice of replete subobjects of $\mathbf{X}$.

3.7 Proposition

Let $\mathcal{G}$ be a trope and $\mathbf{X} \in \mathcal{G}$. Then $\mathcal{F}_{\mathbf{X}}$ is a trope.

The proof is easy and hinges on the fact (3.2) that given $A{:}\mathbf{A} \to \mathbf{X}$, $B{:}\mathbf{B} \to \mathbf{X}$ objects of $\mathcal{F}_{\mathbf{X}}$ and $F{:}A \to B$ then F is a fibration in $\mathcal{F}_{\mathbf{X}}$ iff $F{:}\mathbf{A} \to \mathbf{B}$ is a fibration in $\mathcal{G}$. □

3.8 Proposition

$\mathrm{Fib}_G/\mathbb{C}$ is a trope for any small category $\mathbb{C}$.

There is little left to prove. The terminal object is obviously the identity of $\mathbb{C}$. Let I be a groupoid and $\mathbf{X}{:}\mathbb{X} \to \mathbb{C}$ a fibration of groupoids. A simple calculation will show that the cotensor $\mathbf{X}^I$ in $\mathrm{Fib}_G/\mathbb{C}$ is the fibration $\mathbf{Y}{:}\mathbb{Y} \to \mathbb{C}$ where $\mathbb{Y}$ is the category of all functors $M{:}I \to \mathbb{X}$ such that $\mathbf{X}M$ is a constant functor, and all natural transformations between such functors. There is an obvious "forgetful" functor $\mathbb{Y} \to \mathbb{C}$. Finally, if

$$\begin{array}{ccc} & & \mathbf{E} \\ & & \downarrow E \\ \mathbf{X} & \xrightarrow[F]{} & \mathbf{Y} \end{array}$$

is a diagram in $\mathrm{Fib}_G/\mathbb{C}$, where E is a fibration, we know that E is a fibration as a functor, and then that the pullback $P{:}\mathbb{P} \to \mathbb{X}$ is a Grothendieck fibration. Then defining $\mathbf{P}$ to be the composite $\mathbf{X}P$ makes $\mathbf{P}$ a fibration, since Grothendieck fibrations compose, and therefore $\mathbf{P}$ is thus made an object of $\mathrm{Fib}_G/\mathbb{C}$. It is then trivial to check that the square in $\mathrm{Fib}_G/\mathbb{C}$ thus obtained is a 2-pullback diagram. □

Let us explore the dependent type theory of tropes, which is already quite rich. First we will add the variable declaration $x{\vdots}X$ in contexts to mean that X is a discrete type, interpreted as a discrete fibration. Notice that since discrete fibrations are stable under pullbacks, the structural type formation rules will retain the semantics of $\vdots$. Given a type judgement $[\Gamma]X$ we will use a subscript $[\Gamma]_d X$ to mean that X is a discrete type. Obviously we have the rule

$$\frac{[\Gamma]_d X}{[\Gamma] X}$$

which may or may not be useful. For term judgements we will write $[\Gamma]t{:}X$ to mean that X is discrete.

Tropes give us access to all the structural rules, along with (Sumf), (Pair), (Proj1) and (Proj2) . Since discrete fibrations are closed under composition we have a new rule:

(Sumf$_d$) $$\frac{[\Gamma, x{:}X]_d A}{[\Gamma]_d \Sigma_{x:X} A}$$

Also, the cotensor structure gives us new uniformly defined terms and types. All the discrete types (and only them) have an equality predicate:

(Eq) $$[\Gamma, x{:}X, y{:}X]_p \; x = y$$

We also get internal hom-sets:

(Hom) $$[\Gamma, x{:}X, y{:}X]_d \, \mathrm{Hom}_X(x,y) \quad .$$

We will use the simpler notation $X(x,y)$ for $\mathrm{Hom}_X(x,y)$ whenever it is unambiguous, which is most of the time. Let $\Gamma = x_0{:}X_0, \ldots, x_n{:}X_n$ have the usual interpretation

$$\mathbf{1} \xleftarrow{X_0} \mathbf{X}_0 \xleftarrow{X_1} \mathbf{X}_1 \cdots \xleftarrow{X_n} \mathbf{X}_n$$

and X be interpreted by $X{:}\mathbf{X} \to \mathbf{X}_n$. Therefore $[\Gamma, x{:}X, y{:}Y]$ is interpreted as

$$\mathbf{1} \xleftarrow{X_0} \mathbf{X}_0 \xleftarrow{X_1} \mathbf{X}_1 \cdots \xleftarrow{X_n} \mathbf{X}_n \xleftarrow{X} \mathbf{X} \xleftarrow{X^*X} \mathbf{X} \times_{\mathbf{X}_n} \mathbf{X} \quad .$$

Look at the display map $\langle d_0, d_1 \rangle{:}X^{\mathbf{2}} \to X \times X$ in the trope $\mathcal{F}_{\mathbf{X}_n}$. It translates as a diagram

$$\begin{array}{ccc} \mathbf{Y} & \xrightarrow{\langle d_0, d_1 \rangle} & \mathbf{X} \times_{\mathbf{X}_n} \mathbf{X} \\ & {\scriptstyle X^{\mathbf{2}}} \searrow \quad \swarrow {\scriptstyle X \times X} & \\ & \mathbf{X}_n & \end{array}$$

in $\mathcal{G}$ (here $X \times X$ is not to be confused with the morphism $\mathbf{X} \times \mathbf{X} \to \mathbf{X}_n \times \mathbf{X}_n$, which we will write $X * X$ when we need it to avoid confusion.) But $X \times X$ is obviously $X \circ X^* X$ and we define the interpretation of $Hom_X(x,y)$ to be $\langle d_0, d_1 \rangle$. We also have access to the internal groupoid structure of X : the term judgements

$$[\Gamma, x{:}X] 1_x{:} X(x,x)$$

$$[\Gamma, x{:}X, y{:}X, z{:}X, f{:}X(x,y), g{:}X(y,z)]\, g \circ f{:} X(x,z)$$

have interpretations defined as follows: in the first case, let $i{:}X \to X^2$ be the morphism in $\mathcal{F}_{X_n}$ such that $d_0 \circ i = 1_X = d_1 \circ i$ and ρi is the identity. If $\Delta{:}X \to X \times X$ is the diagonal, we obviously have $i \circ \langle d_0, d_1 \rangle = \Delta$, and so i determines a section i' of $\Delta^* \langle d_0, d_1 \rangle$. But the latter fibration is the interpretation of $Hom_X(x,x)$ and we define the interpretation of 1_x to be i' . In the second case let $\mathcal{T}$ be any trope and $\mathbf{A} \in \mathcal{T}$ an object. The judgement $[x{:}\mathbf{A}, y{:}\mathbf{A}, z{:}\mathbf{A}, f{:}\mathbf{A}(x,y), g{:}\mathbf{A}(y,z)]$ is translated by the sequence

$$\mathbf{1} \xleftarrow{\mathbf{A}} \mathbf{A} \xleftarrow{A_1} \mathbf{A} \times \mathbf{A} \xleftarrow{A_2} \mathbf{A} \times \mathbf{A} \times \mathbf{A} \xleftarrow{A_3} \mathbf{A}^{\to} \times \mathbf{A} \xleftarrow{A_4} \mathbf{B}$$

where A_1 is $\mathbf{A}^* \mathbf{A}$ (that is, the second projection from the product) , $A_2 = (\mathbf{A} \circ A_1)^* \mathbf{A}$, $A_3 = A_2^*(\langle d_0, d_1 \rangle)$, $A_4 = A_3^* C$, $C{:} \mathbf{A} \times \mathbf{A}^{\to} \to \mathbf{A} \times \mathbf{A} \times \mathbf{A}$ being the display map $1_{\mathbf{A}} * \langle d_0, d_1 \rangle$. UGH! WWW.

The reader can verify that all the axioms of a groupoid hold. In particular

$$[\Gamma, x{:}X, y{:}X, \alpha{:}X(x,y)]\ \exists!_{\beta{:}\, X(y,x)} (\alpha \circ \beta = 1_y \wedge \beta \circ \alpha = 1_x)$$

and this gives rise to a term:

$$[\Gamma, x{:}X, y{:}X, \alpha{:}X(x,y)]\ \alpha^{-1}{:}X(y,x) .$$

Also, if $F{:}Y \to X$ is a morphism in $\mathcal{F}/\mathbf{X}_n$, there is $F^2{:}Y^2 \to X^2$ defined by the requirement that $\rho F^2 = F \rho$ (cf 3.1), and this is internalized by the judgement

$$[\Gamma, x{:}Y, y{:}Y, \alpha{:}Y(x,y)] F^2(\alpha){:} X(Fx,Fy) \qquad .$$

It comes as no surprise that the pair (F,F^2) define a functor between the internal groupoids X and Y , as the reader may verify to his leisure. Thus, a trope can truly be seen as a generalization of the category of groupoids.

3.9

Also, if F above is a discrete fibration, let us call W the type that it gives rise to, i.e. $[\Gamma,x:X]_d W(x)$. In this case it is better to view F^2 as a morphism $\langle Fd_0,Fd_1\rangle \to \langle d_0,d_1\rangle = X(-,-)$ in $\mathcal{F}\mathbf{X} \times_{\mathbf{X}_n} \mathbf{X}$: let B be the type $\Sigma_{x:X}W(x)$. Then we have

$$[\Gamma,x,x':X,y:W(x),y':W(x'),\alpha:\mathrm{Hom}_B(\langle x,y\rangle,\langle x',y'\rangle)]\,F^2(\alpha):X(x,x')\ ,$$

and we can internalize our definition of discrete fibration:

$$[\Gamma,x,x':X,y:W(x),\alpha:X(x,x')]\,\exists!_{y':W(x')}\,\exists!_{\beta:B(\langle x,y\rangle,\langle x',y'\rangle)}F^2(\beta)=\alpha\ .$$

Remark

There is more than one notion of groupoid in a trope. We have stressed the idea that "the right" notion is that of an object of $\mathcal{G}$. But for example since $\mathcal{D}_1$ has finite limits we can construct groupoid objects in it. These groupoids are "less internal" than $\mathcal{G}$-groupoids, since they require more information to build than a single object. Eventually they will turn out to be cases of $\mathcal{G}$-*pre*-groupoids.

3.10 Definition

Let $F:\mathbf{F}\to\mathbf{X}$ be a fibration in the trope $\mathcal{G}$. Let

$$\begin{array}{ccc} & Q & \\ \mathbf{P} & \longrightarrow & \mathbf{F} \\ P\downarrow & & \downarrow F \\ \mathbf{X}^2 & \longrightarrow & \mathbf{X} \\ & d_0 & \end{array}$$

be a pullback diagram. A <u>cleavage</u> for E is a pair (Q',β), where $Q':\mathbf{P}\to\mathbf{X}$, $EQ' = d_1P$ and $\beta:Q'\to Q$ is such that $E\beta = \rho P$. Given any fibration the existence of a cleavage is always guaranteed by the definition of a fibration.

3.11 Proposition

Let $\mathcal{E}$ be a category with finite limits. Then the 2-category $\mathrm{Gpd}(\mathcal{E})$ of internal groupoids (also called groupoid objects) in $\mathcal{E}$ is a trope. In particular $\mathcal{E}$ can be $\mathcal{S}et^{\mathbb{C}^{op}}$ where $\mathbb{C}$ is any small category; this is to be compared with the case $\mathrm{Fib}_G/\mathbb{C}$.

In fact, $\mathcal{E}$ could be any category with finite limits. We emphasize the topos case mostly because the internal logic of categories with finite limits [Co] is not as familiar as the internal logic of toposes, although it is a restriction of it. It is easy to see that the 1-category of groupoids in $\mathcal{E}$ has all finite limits, and they are calculated just as for sets; this is because the notion of a groupoid is definable using only finite limits, and limits commute with limits. Let us be a bit more fastidious: let $\mathbb{E}$ be a finitely presented category and $(\mathbf{X}^e)_{e\in\mathbb{E}}$ a diagram $\mathbb{E}\to\mathrm{Gpd}(\mathcal{E})$. That is, for every object $e\in\mathbb{E}$ there is a category object $(X_0^e,X_1^e,d_0^e,d_1^e,i^e,m^e)$ in $\mathcal{E}$ where as usual $d_0^e,d_1^e:X_1^e\to X_0^e$, $i^e:X_0^e\to X_1^e$, $m^e:X_2^e,\to X_1^e$, where X_2^e is the usual pullback, and well-known equations are satisfied, as well as the requirement that $\mathbf{X}^e$ be a groupoid: this can be expressed by a first order sentence in the internal language, and it guarantees the existence of a uniquely defined endomorphism of X_1^e representing the operation of taking the inverse. For every morphism $\sigma:e\to e'$ there are two morphisms X_0^σ,X_1^σ, $X_i^\sigma:X_i^e\to X_i^{e'}$ and they define a functor $\mathbf{X}^\sigma:\mathbf{X}^e\to\mathbf{X}^{e'}$. Then we have just stated that the limit groupoid $\mathbf{Y}$ is constructed by putting the obvious structure on (Y_0,Y_1) where Y_i is $\varprojlim_{e\in\mathbb{E}} X_i^e$. It is easy to verify that $\mathbf{Y}$ is actually the 2-limit of $(\mathbf{X}^e)_e$; recall that given any parallel pair of morphisms $F,G:\mathbf{A}\to\mathbf{B}$ in $\mathrm{Gpd}(\mathcal{E})$ a 2-cell $F\to G$ is a morphism $\alpha:A_0\to B_1$ such that $d_0\alpha=F_0$ and $d_1\alpha=G_0$ (notation follows pattern above.)

So we are left to prove that $\mathrm{Gpd}(\mathcal{E})$ has finite cotensors. It is easy to construct arrow objects: given $\mathbf{X}\in\mathrm{Gpd}(\mathcal{E})$ the $\mathrm{Gpd}(\mathcal{E})$-groupoid $\mathbf{X^2}$ will have for object of objects X_1 and for object of maps the $\mathcal{E}$-set of all commutative squares in $\mathbf{X}$. The internal category structure is given just as if $\mathcal{E}$ were sets. The $\mathcal{E}$-maps $d_0,d_1:X_1\to X_0$ can be naturally extended to morphisms $d_0,d_1:\mathbf{X^2}\to\mathbf{X}$ in $\mathrm{Gpd}(\mathcal{E})$, and there is a natural $\rho:d_0\to d_1$ with the required universal property. Let now I be a free groupoid, generated by the finite graph (G_0,G_1), with $s,t:G_1\to G_0$. To get the cotensor $\mathbf{X^I}$ first construct the category $\mathbb{G}$, whose set of objects is the disjoint union G_0+G_1. The non-identity morphisms of $\mathbb{G}$ are as follows: for every $a\in G_1$ there is a morphism $a^0:a\to s(a)$ and a morphism $a^1:a\to t(a)$. Notice that there are no non-trivial ways to compose morphisms. Let $D:\mathbb{G}\to\mathrm{Gpd}(\mathcal{E})$ be the diagram that sends every $n\in G_0\subset\mathbb{G}$ to $\mathbf{X}$, every $a\in G_1\subset\mathbb{G}$ to $\mathbf{X^2}$, every $a^0:a\to s(a)$ to $d_0:\mathbf{X^2}\to\mathbf{X}$ and every $a^1:a\to t(a)$ to $d_1:\mathbf{X^2}\to\mathbf{X}$. Then it is easy to see that the limit of D will be the desired cotensor $\mathbf{X^I}$. If now I is any finitely presented groupoid, we know by definition that I is the coequalizer of a diagram

$$H \overset{f}{\underset{g}{\rightrightarrows}} K$$

of free groupoids that are both generated by finite graphs. Then taking the equalizer in Gpd($\mathcal{E}$) of

$$\mathbf{X}^K \overset{\mathbf{X}^f}{\underset{\mathbf{X}^g}{\rightrightarrows}} \mathbf{X}^H$$

will give us $\mathbf{X}^I$. We have proved that Gpd($\mathcal{E}$) is a trope, since it has more than the required amount of 2-limits. □

It is natural to ask what a fibration means in Gpd($\mathcal{E}$) , from the point of view of the internal logic of $\mathcal{E}$. The answer is that all fibrations in a trope are cleft (admit a cleavage), and that a cleavage has a precise internal meaning: given a fibration $E{:}\mathbf{E}\to\mathbf{X}$ a cleavage for E is a choice for every $\alpha{:}x'\to x$ in $\mathbf{X}$ and every y in $\mathbf{E}$ above x of a $\beta{:}y'\to y$ above β . In other words since $\mathcal{E}$ does not necessarily have the axiom of choice the naive $\forall\exists$-sentence that defines a fibration is not enough when applied in $\mathcal{E}$ to make E a fibration in Gpd($\mathcal{E}$) ; our definition of a fibration in the latter category requires a choice function.

§ 4 THE HIGHER-ORDER THEORY

First we have to say that the axioms below are not "complete", in the sense that we do not attempt to axiomatize the colimit structure of our model categories. In particular there is no way an analogue of Giraud's theorem can be proved. The reason for limiting ourselves is that some things are not well understood at this time and seem a bit too complicated for comfort. For example categories of fibrations have nice coproducts that are disjoint and universal (these notions make sense since the coprojections are fibrations), but categories of stacks have only bi-coproducts. As we will see still a lot (everything we want to do, actually) can be done foundation-wise. A technical advantage of limiting our ambition is that it makes the proof of theorem 4.16 easy.

We will first give a definition which is slightly too strong, and explore its consequences. We will use the qualifier "strict" to a concept to mean it will be weakened later.

4.1 Definition

A strict universe $\mathcal{G}$ is a trope that satisfies two axioms, the first one being:

SU1 $(\mathcal{G},\mathcal{F})$ admits products. To be more specific, for every fibration $F:\mathbf{Y}\to\mathbf{X}$ the pullback functor F^* has a 2-adjoint Π_F : the usual natural isomorphism $(F^*X,Y)\cong(X,\Pi_F Y)$ is an isomorphism of groupoids, and the Beck condition holds.

Thus we have the rules associated to products: (prodf),(λ-abs) and (eval) . By 2.10 (requiring that all the natural isos of the proof be isos of groupoids) we get that for every $\mathbf{X}\in\mathcal{G}$ the category $\mathcal{F}_{\mathbf{X}}$ is 2-cartesian closed, in the obvious sense, and that for every $\mathcal{G}$-functor $F:\mathbf{Y}\to\mathbf{X}$ the pullback functor preserves the (ordinary!) cartesian closed structure. Now, it is trivial to prove that for every discrete fibration $A:\mathbf{A}\to\mathbf{Y}$, if F above is a fibration then the fibration $\Pi_F A$ is discrete. We thus get a rule for discrete products:

(Prod')
$$\frac{[\Gamma,x:X]_d A}{[\Gamma]_d \Pi_{x:X} A}$$

It follows that (looking at the construction in 2.10) for any B,C objects of $\mathcal{F}_{\mathbf{X}}$, if C is discrete then the exponential object $B \Rightarrow C$ in $\mathcal{F}_{\mathbf{X}}$ is also discrete. In particular $\mathcal{D}_{\mathbf{X}}$, the category of discrete fibrations over $\mathbf{X}$, is cartesian closed and $F^*: \mathcal{D}_{\mathbf{X}} \to \mathcal{D}_{\mathbf{Y}}$ preserves the cartesian closed structure for any $F: \mathbf{Y} \to \mathbf{X}$. The subobjects of the terminator of $\mathcal{D}_{\mathbf{Y}}$ (or of $\mathcal{F}_{\mathbf{Y}}$) are just the replete subobjects of $\mathbf{Y}$. It is easy to show that if $U: \mathbf{U} \to \mathbf{Y}$ is a mono fibration and F a fibration then $\Pi_F U$ is mono too, and so the functor Π_F restricts to a monotone function $\mathcal{R}_{\mathbf{Y}} \to \mathcal{R}_{\mathbf{X}}$ which is the right adjoint to pullback; hence we have the universal quantifier:

$$(\forall f) \qquad \frac{[\Gamma, x{:}X]_p \, \psi}{[\Gamma] \, \forall_{x:X} \, \psi}$$

and it is subject to the rule

$(\forall)$ $\quad [\Gamma] \, \varphi_1, \ldots, \varphi_n \models \forall_{x \in X} \psi \quad$ iff $\quad [\Gamma, x \in X] \, \varphi_1, \ldots, \varphi_n \models \psi \quad$ (x not in Γ)

By 2.10 $\mathcal{R}_{\mathbf{X}}$ is always cartesian closed, and thus we have access to implication:

$$(\Rightarrow f) \qquad \frac{[\Gamma]_p \, \varphi \quad [\Gamma]_p \, \psi}{[\Gamma]_p \; \varphi \Rightarrow \psi} \quad ,$$

subject to the usual rule

$$[\Gamma] \, \varphi_1, \ldots, \varphi_n \models \varphi \Rightarrow \psi \quad \text{iff} \quad [\Gamma] \, \varphi_1, \ldots, \varphi_n, \varphi \models \psi$$

The second axiom is:

SU2 There is a discrete object $\mathbf{\Omega}$ in $\mathcal{G}$ that classifies replete subobjects.

For every $\mathbf{X} \in \mathcal{G}$ we will denote the projection $\mathbf{\Omega} \times \mathbf{X} \to \mathbf{X}$ by $\Omega_{\mathbf{X}}$. Since $\Omega_{\mathbf{X}}$ obviously classifies subobjects in $\mathcal{D}_{\mathbf{X}}$ and $\mathcal{D}_{\mathbf{X}}$ is cartesian closed and has pullbacks, we get that $\mathcal{D}_{\mathbf{X}}$ is an elementary topos. Given $F: \mathbf{Y} \to \mathbf{X}$ it is trivial to show that F^* sends $\Omega_{\mathbf{X}}$ to $\Omega_{\mathbf{Y}}$ and therefore preserves the full topos structure of $\mathcal{D}_{\mathbf{X}}$: F^* is a logical functor.

The immediate syntactical consequence of SU2 is that we can transform predicates into terms: we have the type $\mathbf{\Omega}$ and the constant $T:\Omega$ and given any predicate judgement $[\Gamma]_p\varphi$ there is a characteristic function: $[\Gamma]\chi_\varphi:\Omega$ such that

$$[\Gamma]\varphi \models \chi_\varphi = T \qquad \text{and} \qquad [\Gamma]\chi_\varphi = T \models \varphi \quad .$$

Also, since the morphism $T:\mathbf{1}\to\Omega$ is a discrete fibration, there is a type $[\alpha:\Omega]\,\mathbf{tr}(\alpha)$ which is "monomorphic"

$$[\alpha:\Omega, x:\mathbf{tr}(\alpha), y:\mathbf{tr}(\alpha)]\; x = y$$

this type by itself gives us the comprehension scheme: given $[\Gamma]_p\varphi$ the type $\mathbf{tr}(\chi_\varphi)$ has the same properties as T_φ in 2.13:

$$[\Gamma] \models \varphi \Leftrightarrow \exists!_{x:\mathbf{tr}(\chi_\varphi)} \quad .$$

4.2

The presence of a classifier for repletes also allows us to construct all the other standard logical connectives, by a method due to Prawitz. We refer to [Lk-Sc] for the details. Thus we get the formation rules

$(\vee f)$ $$\frac{[\Gamma]_p\,\varphi \quad [\Gamma]_p\,\psi}{[\Gamma]_p\;\varphi\vee\psi}$$

$(\exists f)$ $$\frac{[\Gamma, x:X]_p\,\varphi}{[\Gamma]\,\exists_{x:X}\,\varphi}$$

(Ff) $\quad []_p\, F$

and the usual rules of intuitionistic logic hold:

$(\vee)$ $\quad [\Gamma]\,\Phi, \varphi\vee\vartheta \models \psi \quad$ iff $\quad [\Gamma]\,\Phi,\varphi \models \psi$ and $[\Gamma]\,\Phi,\vartheta \models \psi$

$(\exists)$ $\quad [\Gamma]\,\Phi, \exists_{x:X}\varphi \models \psi \quad$ iff $\quad [\Gamma, x:X]\,\Phi,\varphi \models \psi$ (x not in Φ)

(F) $\quad [\Gamma]F \models \varphi$

where Φ is short for a list of predicates $\varphi_1,\ldots,\varphi_n$. In what follows we will use the standard notational simplification of identifying a predicate φ with its characteristic function. There is no confusion possible since they live in different worlds. We will be consistent in that approach by considering the logical connectives as operators on Ω when we need to: e.g. if $[\Gamma]\varphi:\Omega$ and $[\Gamma]\psi:\Omega$ then $[\Gamma]\varphi\Rightarrow\psi:\Omega$ is a valid judgement, whose interpretation should be obvious.

Until we say otherwise, $\mathcal{G}$ will denote a strict universe.

4.3 Proposition

For every $\mathbf{X} \in \mathcal{G}$, $\mathcal{F}_{\mathbf{X}}$ is a strict universe.

The proof is easy. □

4.4 Theorem

The inclusion functor $\mathcal{D}_1 \to \mathcal{G}$ has a left adjoint π_0. It obeys the Beck condition in the following way: if $F: \mathbf{Y} \to \mathbf{X}$ is a morphism of $\mathcal{G}$, A an object of $\mathcal{F}_{\mathbf{X}}$ and $\xi: A \to \pi_0(A)$ the universal map in $\mathcal{F}_{\mathbf{X}}$ then $F^*\xi$ is the universal map $F^*A \to F^*(\pi_0(A))$ in $\mathcal{F}_{\mathbf{Y}}$.

Everything we do during this proof belongs to ordinary category theory, not 2-category theory. First notice that the inclusion $\mathcal{D}_1 \to \mathcal{G}$ preserves all finite limits, since it preserves pullbacks and the terminal object. Let $\mathcal{R}(-)$ denote the "internal replete power object functor", i.e. $\mathcal{R}\mathbf{X} = \mathbf{\Omega}^{\mathbf{X}}$. By a very standard argument we know that $\mathcal{R}: \mathcal{G}^{op} \to \mathcal{G}$ has for left adjoint $\mathcal{R}: \mathcal{G} \to \mathcal{G}^{op}$. Therefore we get a monad $(\mathcal{R}^2, \eta, \mu)$ where $\eta: I \to \mathcal{R}^2$ is the adjoint to identity and $\mu = \mathcal{R}\eta\mathcal{R}$. Notice that the image of $\mathcal{R}$ is in $\mathcal{D}_1$ and that if $\mathcal{R}$ is restricted to that topos it is just the ordinary power object functor, and so the monad we get is an extension of the "double power-object monad" $\mathcal{P}^2$, a rather famous one in topos theory [WWW]. If $(\mathbf{X}, h)$ is an $\mathcal{R}^2$-algebra, then $\mathbf{X}$ has to be discrete since it is a retract of a discrete object. Therefore the category $\mathcal{G}^{\mathcal{R}^2}$ of algebras is identical to the category $\mathcal{D}_1{}^{\mathcal{P}^2}$. Let $K: \mathcal{G}^{op} \to \mathcal{G}^{\mathcal{R}^2}$ be the comparison functor, i.e. $K\mathbf{X} = (\mathcal{R}\mathbf{X}, \mathcal{R}\eta\mathbf{X})$. By [B-W,3.3,Prop 4] the diagram

$$\mathcal{R}^5\mathbf{X} \overset{\mathcal{R}\eta\mathcal{R}^2\mathbf{X}}{\underset{\mathcal{R}^3\eta\mathbf{X}}{\rightrightarrows}} \mathcal{R}^3\mathbf{X} \xrightarrow{\mathcal{R}\eta\mathbf{X}} \mathcal{R}\mathbf{X} \qquad (*)$$

is a coequalizer. By a theorem of Paré we know that the restriction of K to $\mathcal{D}_1{}^{op}$ is an equivalence of categories, and so there exists $\pi_0\mathbf{X}$ in $\mathcal{D}_1$ such that $K(\pi_0\mathbf{X}) \cong (\mathcal{R}\mathbf{X}, \mathcal{R}\eta\mathbf{X})$. [B-W,3.3,Theorem10] tells us how to find $\pi_0\mathbf{X}$ by taking the reflexive $\mathcal{R}$-contractible equalizer (coequalizer in $\mathcal{D}_1{}^{op}$) $\varsigma\mathbf{X}$:

$$\mathcal{R}^4\mathbf{X} \overset{\eta\mathcal{R}^2\mathbf{X}}{\underset{\mathcal{R}^2\eta\mathbf{X}}{\leftleftarrows}} \mathcal{R}^2\mathbf{X} \xleftarrow{\varsigma\mathbf{X}} \pi_0\mathbf{X} \qquad (**) .$$

But $\eta\mathbf{X}: \mathbf{X} \longrightarrow \mathcal{R}^2\mathbf{X}$ equalizes $\eta\mathcal{R}^2\mathbf{X}, \mathcal{R}^2\eta\mathbf{X}$, and so there is $\xi\mathbf{X}: \mathbf{X} \to \pi_0\mathbf{X}$ with $\varsigma\mathbf{X} \circ \xi\mathbf{X} = \eta\mathbf{X}$. $(**)$ is in $\mathcal{D}_1$, and so by the Beck theorem, applying $\mathcal{R}$ to it gives a coequalizer diagram in $\mathcal{G}$ (or $\mathcal{D}_1$.) But if we replace $\zeta\mathbf{X}$ by $\eta\mathbf{X}$ in $(**)$ and apply $\mathcal{R}$ we get $(*)$, and since $(*)$ is also a coequalizer, this shows $\mathcal{R}\xi\mathbf{X}$ is an isomorphism. The constructions above are obviously functorial, and we get an endofunctor π_0 of $\mathcal{G}$ along with natural transformations $\xi:\mathrm{Id}\to\pi_0$, $\zeta:\pi_0\to\mathcal{R}^2$ with $\zeta\circ\xi=\eta$, and we claim ξ is a reflector for $\mathcal{D}_1$. Let $\mathbf{A}$ be discrete and $f:\mathbf{X}\to\mathbf{A}$. We want to show there is a unique $g:\pi_0\mathbf{X}\to\mathbf{A}$ such that $g\circ\xi\mathbf{X}=f$. By naturality $\pi_0 f\circ\xi\mathbf{X}=\xi\mathbf{A}\circ f$. By [B-W, 3.3, Lemma 6] and the fact that discrete objects form a topos, in which all monos are regular, we know that since $\mathbf{A}$ is disrete, $\eta\mathbf{A}:\mathbf{A}\to\mathcal{R}^2\mathbf{A}$ is the equalizer of $\eta\mathcal{R}^2\mathbf{A}$ and $\mathcal{R}^2\eta\mathbf{A}$. Since $\zeta\mathbf{A}$ is also the same equalizer, $\xi\mathbf{A}$ has to be an iso, and therefore $(\xi\mathbf{A}^{-1})\circ\pi_0 f$ will fit the bill for g. Now such a g is unique, since $\mathcal{R}\xi\mathbf{X}$ is an isomorphism, and in the topos $\mathcal{D}_1$ $\mathcal{R}$ is faithful. The Beck condition is obvious since the construction of π_0 uses only operations that are preserverved by pullback. □

The syntactical consequence of this is that we have a type formation rule

$$(\pi_0 f)\qquad \frac{[\Gamma]X}{[\Gamma]_d\,\pi_0 X},$$

and the terms

$$[x:X]\,\bar{x}:\pi_0 X$$

$$(\pi_0 f)\qquad \frac{[\Gamma,x:X]\,t:Z}{[\Gamma,y:\pi_0(X)]\,(\tau xy:X):Z},$$

where τ is an operator that binds the first variable after it like λ (x in this case), but such that a new free variable (y in this case) always needs to be introduced. The adjunction gives us the external equations

$$[\Gamma,x:X]((\tau xy:X)t)[y\vee\bar{x}]\equiv t$$

WWW

4.5 Note

Given any fibration $F:\mathbf{Y}\to\mathbf{X}$ the pair (Π_F,F^*) can be considered as a geometric morphism of toposes $\mathcal{D}_\mathbf{Y}\to\mathcal{D}_\mathbf{X}$; since F^* is logical such a geometric morphism is called an atomic morphism of toposes. For more on atomic morphisms the reader is referred to Barr-Diaconescu [B-D] ; they are closely related to groupoids in the target topos ($\mathcal{D}_\mathbf{X}$ here.) The reader should agree that our axioms are a natural generalization of those for an elementary topos. In fact, a strict universe all whose objects are discrete is an elementary topos, and vice-versa. There is one essential difference with classical topos theory, though: the Lawvere-Tierney axioms ensure all finite colimits, and this is not the case here, as we have said above.

4.6 Theorem

Let $\mathcal{E}$ be a topos. Then Gpd($\mathcal{E}$) , the 2-category of all groupoid objects in $\mathcal{E}$, is a strict universe. In particular, Gpd(*Set*) , the category of ordinary groupoids, is one.

WWW . □

4.7 Remark

In the category of groupoids (or any category of the form Gpd($\mathcal{E}$) , for that matter), given *any* morphism $F:\mathbf{X}\to\mathbf{Y}$, the pullback functor $F^*:\mathcal{D}_\mathbf{Y}\to\mathcal{D}_\mathbf{X}$ will have both a left and a right adjoint. This is simply the theory of Kan extensions; these adjoints do not seem to be a consequence of our axiomatization. But notice that the Beck condition will not be true in general: for an easy counterexample just take the pullback square

$$\begin{array}{ccc} \varnothing & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ \{1\} & \longrightarrow & \{0,1\} \end{array}$$

where all the arrows are inclusions and the set $\{0,1\}$ is given the full equivalence relation $\{0,1\}\times\{0,1\}$. We conclude from this that Kan extensions in general do not have syntactical meaning, at least not at this present stage of our knowledge of syntax. Hence the extra Kan extensions are unessential features of our models, inasmuch as these models are to be thought of as interpretations of syntactical entities.

We have said that the notion of strict universe is a bit too strong for our needs. The reason is that there is a way in which a 2-

category of fibrations admits products, but it is not the strict way that we have described above; we have to resort to some "bi-categorical" notions.

4.7 Proposition

Let $K:J\to I$ be a functor between ($\mathcal{Set}$-)groupoids. Then the following are equivalent:

i) K is surjective on objects and an equivalence of groupoids

ii) K is full and faithful and surjective on objects.

iii) K is a fibration such that every fiber is equivalent to the one-element groupoid.

The proof of i)⇒ii) is trivial since an equivalence is always full and faithful. To prove ii)⇒iii) let $j\in J$ and $\alpha:i'\to Kj$. Since K is surjective there exists $j'\in J$ with $Kj'=i'$. Since K is full and faithful there exists a unique $\beta:j'\to j$ with $K\beta=\alpha$ and therefore K is a fibration. Since K is full and faithful for every $k,k'\in J$ such that $Kk=Kk'$ there exists a unique $\gamma:k\to k'$ above identity, and so every fiber is equivalent to the one-element groupoid. We leave the proof of iii)⇒i) to the reader.

We will call a morphism of groupoids satisfying the conditions above a <u>surjective</u> <u>equivalence</u> .

4.9 Definition

Let $\mathcal{A},\mathcal{B}$ be 2-categories, $F:\mathcal{A}\to\mathcal{B}$ a 2-functor. We say F <u>has a</u> <u>right</u> <u>bi-adjoint</u> [WWW] if for every $X\in\mathcal{B}$ there exists an object $UX\in\mathcal{A}$ and a morphism $\varepsilon X:FUX\to X$ such that for every $Y\in\mathcal{A}$ the functor "apply F and postcompose with εX": $\mathcal{A}(Y,UX)\to\mathcal{B}(FY,X)$ is an equivalence of categories. We say F <u>has</u> <u>a</u> <u>loose</u> <u>right</u> <u>adjoint</u> if the functor defined above is a surjective equivalence. The corresponding "left" notion is defined by duality.

Contrarily to ordinary adjoints, U is not defined up to isomorphism but only up to equivalence, and does not extend to a 2-functor $\mathcal{B}\to\mathcal{A}$ but only to a pseudo-functor.

4.10 Definition

Let $\mathcal{G}$ be a trope. We say $\mathcal{G}$ <u>admits</u> <u>loose</u> <u>products</u> if for every pair of fibrations $F:\mathbf{F}\to\mathbf{Y}$ the pullback functor $\mathcal{F}_{\mathbf{Y}}\to\mathcal{F}_{\mathbf{X}}$ has a loose right adjoint: for every $A:\mathbf{A}\to\mathbf{F}$, there is a fibration $\Pi_F A:\Pi_F\mathbf{A}\to\mathbf{Y}$ and a morphism $ev:F^*\Pi_F A\to A$ in $\mathcal{F}_{\mathbf{F}}$ such that for every fibration $B:\mathbf{B}\to\mathbf{Y}$ the operation "pull back by F and postcompose by ev " induces a morphism of groupoids $\mathcal{F}_{\mathbf{Y}}(B,\Pi_F A)\to\mathcal{F}_{\mathbf{F}}(F^*B,A)$ which is a surjective equivalence, and if the Beck condition holds in the following sense: for every pullback square

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{S} & \mathbf{F} \\ E\downarrow & & \downarrow F \\ \mathbf{X} & \xrightarrow[X]{} & \mathbf{Y} \end{array}$$

where E,F are fibrations and every $A:\mathbf{A}\to\mathbf{F}$ in $\mathcal{F}_F$ the pair

$$(\ X^*\Pi_F A\ ,\ E^*X^*\Pi_F A = S^*F^*\Pi_F A \xrightarrow{S^*(ev)} S^*A\ )$$

has the universal property described above with A replaced by S^*A , i.e. it induces a surjective equivalence $\mathcal{F}_{\mathbf{X}}(C,X^*\Pi_F A)\to\mathcal{F}_{\mathbf{E}}(E^*C,S^*A)$.

Our insistence on surjective equivalences makes things a bit stricter (and much less complicated) than the standard theory of bi-adjoints. But naturally, as we have said above, given A,F as above $\Pi_F A$ is defined only up to equivalence in $\mathcal{G}$, not isomorphism, as the reader can verify. If we choose a value of $\Pi_F A$ for every fibration $F:\mathbf{F}\to\mathbf{Y}$ in $\mathcal{G}$ and every $A\in\mathcal{F}_{\mathbf{F}}$ we get a form of lambda calculus where currying is defined "up to a unique 2-cell". For example, given $A_i\in\mathcal{F}_{\mathbf{F}}$, $i=1,2,3$, choose a value for $\Pi_F A_i$. Then, for $S:A_1\to A_2$, $T:A_2\to A_3$ we can choose values for $\Pi_F S:\Pi_F A_1\to\Pi_F A_2$, $\Pi_F T:\Pi_F A_2\to\Pi_F A_3$, and $\Pi_F(TS):\Pi_F A_1\to\Pi_F A_3$. Then, it not guaranteed that $\Pi_F(TS)=\Pi_F T\circ\Pi_F S$, but there will be a uniquely defined "natural" isomorphism $\Pi_F(TS)=\Pi_F T\circ\Pi_F S$. This would seem to wreak havoc with the syntax, but we will see that almost everything can be salvaged.

4.11 Definition

A <u>universe</u> is a trope $\mathcal{G}$ such that

U1 $\mathcal{G}$ admits loose products.

U2 There is a discrete object $\mathbf{\Omega}$ in $\mathcal{G}$ that classifies replete subobjects.

Now we can go through all the facts proven in 4.1 and 4.2 and see that they also apply to the case of a non-strict universe. In particular, if $\mathcal{G}$ is a universe then $\mathcal{F}_{\mathbf{X}}$ is a universe for any object $\mathbf{X}$. Also, since a surjective equivalence of discrete groupoids is an isomorphism of sets, the adjunctions dealing with discrete fibrations are ordinary adjunctions:

4.12 Proposition

Let $\mathcal{G}$ be a universe. Then

i) If $F:\mathbf{F}\to\mathbf{X}$ is a fibration and $A:\mathbf{A}\to\mathbf{F}$ a discrete fibration then $\Pi_F A$ is a discrete fibration and satisfies the *isomorphism* $\mathcal{F}_{\mathbf{F}}(F^*B,A) \cong \mathcal{F}_{\mathbf{X}}(B,\Pi_F A)$ for any $B\in\mathcal{F}_{\mathbf{X}}$. For any $C\in\mathcal{F}_{\mathbf{F}}$ the strict exponential object $C\Rightarrow A$ exists and is discrete.

ii) For any $\mathbf{X}\in\mathcal{G}$ the category $\mathcal{D}_{\mathbf{X}}$ is an elementary topos, and pullback functors preserve the full topos structure. Pulling back by a fibration has both a left and a right adjoint, and the ordinary Beck condition holds. □

We now have to show how the type theory is adapted to non-strict universes. Recall that the full interpretation of a term judgement $[\Gamma]t{:}Y$ is a diagrams $D\to\mathcal{G}$ whose auxiliary part guarantees it is only defined up to unique isomorphism. But now we are in a 2-category, and we can extend the uniqueness-up-to-unique-isomorphism to *terms*: that is, in the next paper of the series we will define a notion of 2-graph which will have both an "interesting" and an auxiliary part, and such that its possible interpretations will be 2-diagrams that are defined in a suitable unique way. For this to work it is necessary to restrict the (NwCn) rule as follows: it can only be applied to judgements $[\Gamma]Y$ such that the Π operator *does not appear anywhere in* Γ *or* Y.

4.13 Theorem

Let $\mathbb{C}$ be a category. Let $\mathcal{G}$ be $\mathrm{Fib}_G/\mathbb{C}$. Then $\mathcal{G}$ is a universe.

Remember that for any object $\mathbf{X}:\mathbb{X}\to\mathbb{C}$ of $\mathrm{Fib}_G/\mathbb{C}$ the 2-category $\mathcal{F}_{\mathbf{X}}$ is equivalent, in the strictest possible sense of equivalence, to the 2-category $\mathrm{Fib}_G/\mathbb{X}$. To avoid an orgy of symbols, we will often identify these 2-categories, and say things like: let $F:\mathbb{F}\to\mathbb{X}$ be a fibration above $\mathbf{X}$. In other words we have a diagram

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{F} & \mathbb{X} \\ {\scriptstyle \mathbf{F}}\searrow & & \swarrow{\scriptstyle \mathbf{X}} \\ & \mathbb{C} & \end{array}$$

of categories where every functor is a fibration. Given $F, \mathbb{F}, \mathbf{F}$ as above, let $A: \mathbf{A} \to \mathbf{F}$ be in $\mathcal{F}_{\mathbf{F}}$, i.e. $A: \mathbb{A} \to \mathbb{F}$ in $\mathrm{Fib}_{\mathbb{G}}/\mathbb{F}$. The object $\Pi_F A: \mathbb{E} \to \mathbb{X}$ will be constructed as follows: let $x \in \mathbb{X}$ be above $S \in \mathbb{C}$. an object of $\mathbb{E}$ above x is a pair (γ, R), where

- $\gamma: \mathbb{C}/S \to \mathbb{X}$ is a morphism $U_S \to \mathbf{X}$ in $\mathcal{G}$ such that $\gamma(1_S) = x$ (U_S as in 3.4) ; in other words γ is a choice of a cartesian arrow to x for every morphism to S.
- R "would be a morphism $F^*\gamma \to A$ in $\mathcal{F}_{\mathbf{F}}$ if γ were a fibration" ; in other words, R is a functor from the pullback object $F^*(\mathbb{C}/S)$ making the triangle commute:

$$\begin{array}{ccc} & \mathbb{A} & \\ {\scriptstyle R}\nearrow & & \searrow{\scriptstyle A} \\ F^*(\mathbb{C}/S) & \to & \mathbb{F} \\ \downarrow & & \downarrow{\scriptstyle F} \\ \mathbb{C}/S & \xrightarrow[\gamma]{} & \mathbb{X} \end{array}$$

To describe the universal arrow $ev: F^*\Pi_F A \to A$, notice that an object of

$$\begin{array}{ccc} F^*\mathbb{E} & \xrightarrow{ev} & \mathbb{A} \\ {\scriptstyle F^*\Pi_F A}\searrow & & \swarrow{\scriptstyle A} \\ & \mathbb{F} & \end{array}$$

$F^*\mathbb{E}$ above $y \in \mathbb{F}$ is an object of $\mathbb{E}$ above Fy (and and that that is, it is a pair (γ, R), where WWW . □

We can now give a synthetic approach to the theory of stacks. As usual, given a universe $\mathcal{G}$, a topology on $\mathcal{G}$ is a $j: \mathbf{\Omega} \to \mathbf{\Omega}$ satisfying $[\varphi: \mathbf{\Omega}] \models j(j(\varphi)) = j(\varphi)$, $[\varphi: \mathbf{\Omega}] \models \varphi \Rightarrow j(\varphi)$, $[\varphi, \psi: \mathbf{\Omega}] \models j(\varphi \wedge \psi) = j(\varphi) \wedge j(\varphi)$. Given $\mathbf{A} \in \mathcal{G}$ and a replete $\mathbf{B} \hookrightarrow \mathbf{A}$, the closure $\overline{\mathbf{B}} \hookrightarrow \mathbf{A}$ is defined as usual, and $\mathbf{B}$ is said to be dense if $\overline{\mathbf{B}} = \mathbf{A}$, saturated if $\overline{\mathbf{B}} = \mathbf{B}$.

4.15 Definition

Let j be a topology on $\mathcal{G}$. A j-stack is an object $\mathbf{X} \in \mathcal{G}$ such that for every dense $\mathbf{B} \to \mathbf{A}$ the induced functor $\mathcal{G}(\mathbf{A}, \mathbf{X}) \to \mathcal{G}(\mathbf{B}, \mathbf{X})$ is an *equivalence* of goupoids. The 2-full subcategory of $\mathcal{G}$ whose objects are the j-stacks is denoted $St_j(\mathcal{G})$.

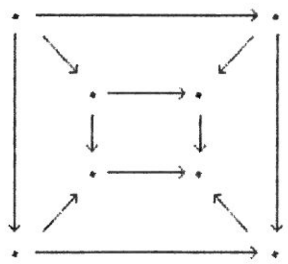
4.16 Theorem

For every j $St_j(\mathcal{G})$ is a universe. If $\mathcal{G}$ is strict, $St_j(\mathcal{G})$ is strict too.

The first part of the proof is to show that $St_j(\mathcal{G})$ is a trope. We first show that $St_j(\mathcal{G})$ is closed under, and therefore has, finite cotensors: if $\mathbf{X}$ is a j-stack, I a finitely presented groupoid, $\mathbf{B}\to\mathbf{A}$ a dense replete mono, then since the bottom row is obviously an equivalence and the sides isomorphisms

$$\begin{array}{ccc} \mathcal{G}(\mathbf{A},\mathbf{X}^I) & \longrightarrow & \mathcal{G}(\mathbf{B},\mathbf{X}^I) \\ \downarrow & & \downarrow \\ Gpd(I,\mathcal{G}(\mathbf{A},\mathbf{X})) & \longrightarrow & Gpd(I,\mathcal{G}(\mathbf{B},\mathbf{X})) \end{array}$$

we get that the top row is an equivalence, QED. Then we show that if $\mathbf{F}\to\mathbf{X}$ is a morphism of j-stacks which is a fibration in $\mathcal{G}$, its pullback in $\mathcal{G}$ by any morphism $\mathbf{Y}\to\mathbf{X}$ of j-stacks is is a morphism of j-stacks. The least offensive way we have found of proving this is to first show the following lemma: if



is a commutative cube of *Set*-groupoids (all faces commute), where all the vertical arrows are fibrations, both left and right faces pullbacks, and all horizontal arrows but the top one equivalences, then the top horizontal arrow is an equivalence. The proof will be left to the reader (hint: recal that a morphism $F{:}I\to J$ of groupoids is an equivalence iff it is full and faithful, and for every $j\in J$ there is $i\in I$ such that $Fi\cong j$.) Then given a dense $\mathbf{B}\to\mathbf{A}$ we just apply this lemma to the cube whose left face is $\mathcal{G}(\mathbf{A},$ the pullback square$)$, whose right face is $\mathcal{G}(\mathbf{B},$ the pullback square$)$ (keeping the fibrations vertical!), and whose horizontal arrows are induced by $\mathbf{B}\to\mathbf{A}$.

All we have left to do to prove $St_j(\mathcal{G})$ is a trope is to show that a fibration in $St_j(\mathcal{G})$ is a fibration in $\mathcal{G}$. But if $F{:}\mathbf{F}\to\mathbf{X}$ is a fibration in $St_j(\mathcal{G})$ it admits a cleavage (Q',β) (these symbols, along with $\mathbf{P}$

have the same meaning as in 3.10) in $St_j(\mathcal{G})$ because $\mathbf{P} \in St_j(\mathcal{G})$ since $d_0: \mathbf{X^2} \to \mathbf{X}$ is a morphism of stacks, and a fibration in $\mathcal{G}$, by 3.2 c). We then use that cleavage to show F is a fibration in $\mathcal{G}$, since it will allow us to lift anything we want.

We now want to show $St_j(\mathcal{G})$ admits loose products, or the stricter variety if $\mathcal{G}$ is strict. All we have to do is show that if $F: \mathbf{F} \to \mathbf{X}$ and $A: \mathbf{A} \to \mathbf{X}$ are fibrations of stacks, then $\Pi_F A$ is a fibration of stacks, and this is very easy.

Finally, let $\mathbf{\Omega}_j$ be the equalizer of j and identity. $\mathbf{\Omega}_j$ is obviously the classifier of saturated replete subobjects. It is left to show that given a sheaf $\mathbf{X}$ and a replete subobject $\mathbf{Y} \hookrightarrow \mathbf{X}$, $\mathbf{Y}$ is a subsheaf iff $\mathbf{Y}$ is saturated. WWW . □

4.16 Remark

If $\mathcal{G}$ is $Fib_G/\mathbb{C}$, where $\mathbb{C}$ has finite limits, then a j-stack in our definition is the same as a stack in the traditional definition [Gi] (we are very happy not to have to use the traditional definition in this paper.) If $\mathcal{G}$ is $Gpd(\mathcal{E})$ for a topos $\mathcal{E}$ then a j-stack is something new, which we think has never been explored. Naturally, another way to get a universe from $\mathcal{E}$ and j is to take the category $Gpd(Sh_j(\mathcal{E}))$ [proof that they are not "equivalent"WWW] . A conclusion of this is that the notion of universe is a more subtle one, in a sense, than the notion of elementary topos, since there is more than one way of embedding a topos $\mathcal{E}$ in a universe $\mathcal{G}$ in such a way that $\mathcal{E}$ is the topos of discrete objects of $\mathcal{G}$ (also, compare $Gpd(\mathcal{S}et^{\mathbb{C}^{op}})$ and $Fib_G/\mathbb{C}$.)

It seemed at first that the "right" definition of stack would have used surjective equivalences instead of ordinary equivalences of groupoids, but it seems impossible to prove that the categories of stacks thus defined are tropes in general. Both notions coincide if $\mathcal{G}$ is the paradigmatic $Fib_G/\mathbb{C}$.

§5 KRIPKE-JOYAL SEMANTICS

Given $\mathbf{X} \in \mathcal{G}$ we will sometimes use the notation $\mathbf{X}$ for the full subobject $\mathbf{X} \hookrightarrow \mathbf{X}$, and sometimes $\top$, when the context is clear. Also we will use $F_{\mathbf{X}} \hookrightarrow \mathbf{X}$ to denote the least replete subobject of $\mathbf{X}$. $F_{\mathbf{X}}$ is the subobject classified by $F \circ ! : \mathbf{X} \to \Omega$ where F is falsehood. Let us recall that a predicate judgement

$$[x_0 : X_0, \dots, x_n : X_n]_p \ \varphi$$

is interpreted as a sequence

$$\begin{array}{ccccc} & X_0 & X_1 \ \cdots & X_{n-1} & X_n \\ \mathbf{1} \longleftarrow & \mathbf{X}_0 \longleftarrow & \mathbf{X}_1 \dots \longleftarrow & \mathbf{X}_{n-1} \longleftarrow & \mathbf{X}_n \hookleftarrow [\![\varphi]\!] \end{array}$$

in $\mathcal{G}$, the subobject at the end being replete. Given a judgement as above we define $[\![\varphi]\!]^{\#}$ to be the set

$$[\![\varphi]\!]^{\#} = \{ (\mathbf{I}, a), a : \mathbf{I} \to \mathbf{X}_n \mid a \text{ factors through } [\![\varphi]\!] \}$$

5.1 Definition

Given $\mathbf{X} \in \mathcal{G}$ a <u>covering family</u> of $\mathbf{X}$ is a finite family $(Y_i : \mathbf{Y}_i \to \mathbf{X})_i$ of fibrations such that the sup over i of the subobjects $\exists_{Y_i}(\top)$ is $\mathbf{X}$, in other words such that

$$[x : \mathbf{X}] \models \bigvee_i \left(\exists_{y_i \in Y_i(x)} \top \right)_i \qquad .$$

(Remember: the instances of $\top$ are there only to make the syntax correct!.) A one-element covering family is called a <u>surjection</u> . Obviously, a $\mathcal{G}$-groupoid that admits the empty family as a covering is an object that has only itself as a replete subobject. In all the models we know the only such object is the initial one $\mathbf{0}$, but we cannot prove it is always this way.

5.2 Proposition

Let $F : \mathbf{F} \to \mathbf{X}$ be a fibration and $\mathbf{Y} \hookrightarrow \mathbf{X}$ replete. Then the universal morphism $\mathbf{Y} \to \exists_F \mathbf{Y}$ is a surjection. Surjective families are stable under pullback and composition (the meaning of "stability under composition" for covering families being as usual.)

The proofs are easy. □

5.3 Theorem

Let Γ be the judgement $x_0:X_0, \ldots, x_n:X_n$ and let it be interpreted by

$$\begin{array}{ccccc} X_0 & X_1 & \cdots & X_{n-1} & X_n \\ \mathbf{1} \longleftarrow \mathbf{X}_0 \longleftarrow & \mathbf{X}_1 & \ldots \longleftarrow & \mathbf{X}_{n-1} \longleftarrow & \mathbf{X}_n \end{array} .$$

Let $[\Gamma]_p \psi$, $[\Gamma]_p \theta$ be predicate judgements, and $a:\mathbf{I}\to\mathbf{X}_n$, $c:\mathbf{I}\to\mathbf{X}_{n-1}$. Then

i) $a \in [\![T]\!]^\#$ always.

ii) $a \in [\![F]\!]^\#$ iff $\mathbf{I}$ admits the empty cover.

iii) $a \in [\![\psi \wedge \theta]\!]^\#$ iff $a \in [\![\psi]\!]^\#$ and $a \in [\![\theta]\!]^\#$.

iv) $a \in [\![\psi \Rightarrow \theta]\!]^\#$ iff for every $f:\mathbf{J}\to\mathbf{I}$, if $af \in [\![\psi]\!]^\#$ then $af \in [\![\theta]\!]^\#$.

v) $a \in [\![\psi \vee \theta]\!]^\#$ iff there exists a covering family $(f_i:\mathbf{J}_i\to\mathbf{I})_i$, along with morphisms $(b_i:\mathbf{J}_i\to\mathbf{X}_n)_i$ such that for every i , either $b_i \in [\![\psi]\!]^\#$ or $b_i \in [\![\theta]\!]^\#$.

vi) $c \in [\![\forall_{x_n:X_n} \psi]\!]^\#$ iff for every $f:\mathbf{J}\to\mathbf{I}$ and $b:\mathbf{J}\to\mathbf{X}_n$ such that $X_n \circ a = cf$ we have $b \in [\![\psi]\!]^\#$.

vii) $c \in [\![\exists_{x_n:X_n} \psi]\!]^\#$ iff there exists a surjection $f:\mathbf{J}\to\mathbf{I}$ and a morphism $b:\mathbf{J}\to\mathbf{X}$ with $X_n \circ b = cf$ such that $b \in [\![\psi]\!]^\#$.

The proof is straighforward, just like [Lk-Sc]. □

Kripke-Joyal semantics does not have to be done all over $\mathcal{G}$, if we are lucky:

5.4 Definition

A <u>generating class</u> $\mathfrak{C}$ (there should be a qualifier here, since there are so many kinds of generators...) is a class of objects of $\mathcal{G}$ such that given a replete mono $\mathbf{A}\to\mathbf{B}$ in $\mathcal{G}$ it is an iso iff $\mathcal{G}(\mathbf{G},\mathbf{X})\to\mathcal{G}(\mathbf{G},\mathbf{Y})$ is an iso for all $\mathbf{G} \in \mathfrak{C}$.

In all the models we have seen so far (in all the models we know!) $\mathcal{D}_1$ is a generating class. Also, if $\mathcal{G}$ is $\mathrm{Fib}_G/\mathbb{C}$, the small subclass of $\mathcal{D}_1$ of all representables is a generating class. The point is that given $[\![\varphi]\!]\to\mathbf{X}$, in order to define $[\![\varphi]\!]^\#$ we can restrict ourselves to morphisms $\mathbf{G}\to\mathbf{X}$ where $\mathbf{G}$ is in $\mathfrak{C}$; there are enough such morphisms to recover $[\![\varphi]\!]$, and the definition of the semantics will apply just the same.

5.5 Definition

Let $[\Gamma,x{:}X]_p\varphi$ be a predicate judgement. We use the notation $\exists!!_{x:X}\varphi$ for $\exists_{x:X}\varphi$ to mean that it is also true that $[x{:}X,y{:}X] \models \exists!_{\alpha:X(x,y)}\top$. $\exists!!$ is the predicate of existence up to unique isomorphism, the most important connective in category theory.

Some of our models are distinguished by an important property:

5.6 Definition

We say $\mathcal{G}$ has the <u>axiom of trivial choice</u> (ATC) if every surjective equivalence in $\mathcal{G}$ splits; that is, given a fibration $Y(-){:}\,\mathbf{Y}\to\mathbf{X}$ such that

$$[x{:}\mathbf{X}] \models \exists!!_{y:Y(x)}\top$$

then Y has a splitting, in particular (modulo the restrictions on (NwCn) in 4.12) there exists a term t such that

$$[x{:}\mathbf{X}] \models t(x){:}\,Y(x)\ .$$

5.7 Proposition (parametrized form of ATC)

Let $\mathcal{G}$ have ATC. Let

$$[\Gamma]Y$$

be a type judgement such that

$$[\Gamma] \models \exists!!_{y:Y} \qquad , \qquad (*)$$

Then (modulo...) there exists a term t with

$$[\Gamma]\,t{:}Y \qquad .$$

The proof is just the verification that in the interpretation of the judgement $[\Gamma]Y$, $(*)$ means that

$$\begin{array}{ccccc} X_0 & X_1 & \cdots & X_n & Y \\ \mathbf{1}\longleftarrow\mathbf{X}_0\longleftarrow & \mathbf{X}_1 \ldots & \mathbf{X}_{n-1}\longleftarrow & \mathbf{X}_n\longleftarrow & \mathbf{Y} \end{array}$$

the fibration Y is a surjective equivalence. □

5.8 Proposition (internal form of ATC)

Let $\mathcal{G}$ have ATC . Then for any type $[\Gamma,x{:}X]\,Y$ we have

$$[\Gamma] \models \forall_{x:X} \exists!!_{y:Y} \Rightarrow \exists!!_{f:\Pi_{x:X} Y} \top \quad .$$

WWW . □

5.9 Proposition

For any small category $\mathbb{C}$, $Fib_G/\mathbb{C}$ has ATC.

This is just because a surjective equivalence $\mathbf{Y} \to \mathbf{X}$ in $Fib_G/\mathbb{C}$ is a surjective equivalence $\mathbb{Y} \to \mathbb{X}$ of categories in our world, and surjective equivalences split in our world.

5.10 Theorem

Let $(\mathbb{C},j)$ be a subcanonical site. Then $St_j(Fib_G/\mathbb{C})$ admits ATC.

The proof will make use of the fact [Ba1] that in a subcanonical site one-morphism covering families are regular epis, and that the inclusion $\mathcal{D}_1 \to \mathcal{G}$ in a category of fibrations "preserves finite colimits", in the sense that every finite colimit diagram in $\mathcal{D}_1$ is a loose colimit diagram in $\mathcal{G}$. WWW . □

The axiom of trivial choice has very significant consequences for the foundations of category theory in a topos. It is a form of choice which is available in universes whose logic can very well be non-boolean; it basically says that "existence up to unique isomorphism always defines a function". Naturally, the quantifier of existence up to unique isomorphism is THE important quantifier in the universal algebra of categories; this means that category theory in a universe with ATC becomes very much like category theory in our world of sets with choice (the only world so far where category theory is a comfortable activity), provided we take care of defining the categories in such a way that their isomorphisms are the internal isomorphisms of the universe; this is the subject of the next chapter. For example, if we take the notion of a category with products, we can define it in a universe in such a way that the two conflicting constructive definitions, "weak" and "strong", i.e.

For every pair of objects there exists a product diagram

and

There exists a functor $\mathbf{C}\times\mathbf{C}\to\mathbf{C}$ along with natural transformations blablabla

actually coincide.

§6 CATEGORY THEORY

6.1 Definition

A proto-category $\mathbf{C}$ in $\mathcal{G}$ (or $\mathcal{G}$-proto-category) is a quadruple $(|\mathbf{C}|, Hom_{\mathbf{C}}(-,-), I_{(-)}, (-)\diamond(-))$, where (for ease of notation we will write $\mathbf{C}(-,-)$ for $Hom_{\mathbf{C}}(-,-)$ whenever possible)

a) $|\mathbf{C}| \in \mathcal{G}$ and $\mathbf{C}(-,-): \mathbf{C}^{\rightarrow} \rightarrow |\mathbf{C}| \times |\mathbf{C}|$ is a *discrete* fibration, i.e. $[x,y:|\mathbf{C}|]_d\, \mathbf{C}(x,y)$.

b) $[x:|\mathbf{C}|]\, I_x : \mathbf{C}(x,x)$.

c) $[x,y,z:|\mathbf{C}|, f:\mathbf{C}(x,y), g:\mathbf{C}(y,z)]\, g\diamond f:\mathbf{C}(x,z)$

d) The usual associativity and unit properties hold.

e) An additional equation holds, which will be described shortly.

In other words a)-d) just say that a proto-category is an ordinary category object in $\mathcal{G}$, requiring that the usual $d_0,d_1:\mathbf{C}_1 \rightarrow \mathbf{C}_0$, here denoted $\mathbf{C}(-,-): \mathbf{C}^{\rightarrow} \rightarrow |\mathbf{C}| \times |\mathbf{C}|$ be a discrete fibration, so we can have access to the internal predicate of equality between parallel arrows, and thus internalize everything.

From now on the variable declaration $x:|\mathbf{C}|$ will be written $x:\mathbf{C}$, as we have always done in the metalanguage. In order to describe the last equation we need a notation for the internal presheaf $\mathbf{C}(-,-)$'s action on internal morphisms of $|\mathbf{C}|$. We will consider $\mathbf{C}(-,-)$ as a left-contravariant, right-covariant functor, as befits a hom-set functor:

$$[c,c',d,d':\mathbf{C}, \alpha:|\mathbf{C}|(c',c), \beta:|\mathbf{C}|(d,d'), f:\mathbf{C}(c,d)]\ \beta \# f \# \alpha : \mathbf{C}(c',d') .$$

In other words the ternary operation $(-)\#(-)\#(-)$ is obtained by applying the method of 3.9, mutatis mutandi. It goes without saying that

$$[\ldots] \models 1_d \# f \# 1_c = f \ \wedge\ \beta' \# (\beta \# f \# \alpha) \# \alpha' = (\beta' \circ \beta) \# f \# (\alpha \circ \alpha') \quad .$$

We abbreviate $\beta \# f \# 1_c$ by $\beta \# f$ and $1_d \# f \# \alpha$ by $f \# \alpha$. Condition e) is that the "Yoneda" operation $[c,d:\mathbf{C}, \alpha:|\mathbf{C}|(c,d)] \models \alpha \# I_c : \mathbf{C}(c,d)$ is a functor, i.e.

$$[c,d,e:\mathbf{C}, \alpha:|\mathbf{C}|(c,d), \beta:|\mathbf{C}|(d,e)] \models (\beta \# I_d) \diamond (\alpha \# I_c) = (\beta \circ \alpha) \# I_c \ .$$

For example, an object $\mathbf{X}$ of $\mathcal{G}$ always has a natural proto-category structure, given by taking $f \diamond g = f \circ g$. Given proto-categories $\mathbf{C}, \mathbf{D}$ a

<u>functor</u> $F:\mathbf{C}\to\mathbf{D}$ is defined just as usual, i.e. it is a pair (F,F_1) where $F:|\mathbf{C}|\to|\mathbf{D}|$ and

$$[x,y:|\mathbf{C}|,f:\mathbf{C}(x,y)]F_1f:\mathbf{D}(Fx,Fy)\ , \qquad \text{satisfying the usual}$$

$$[x,y\ldots]\models F_1(g\diamond f)=F_1f\diamond F_1g\wedge F_1(I_x)=I_{Fx}\quad .$$

Given a functor (F,F_1) as above we write Ff for F_1f, to conform to usage.

From now on we will denote $f\diamond g$ by $f\circ g$ and I_c by 1_c. Let Iso(f) denote the predicate " f is an isomorphism" i.e.

$$[x,y:\mathbf{C},f:\mathbf{C}(x,y)]\models \mathrm{Iso}(f)\Leftrightarrow(\exists_{g:\mathbf{C}(y,x)}\ f\circ g=1_x\wedge g\circ f=1_y)$$

notice that we always have, by the functoriality of "Yoneda"

$$[x,y:\mathbf{C},\alpha:|\mathbf{C}|(x,y)]\models \mathrm{Iso}(\alpha\# I_x)\ .$$

A $\mathcal{G}$-<u>pre</u>-<u>category</u> is a proto-category $\mathbf{C}$ such that "Yoneda" is injective, i.e.

$$[x,y:\mathbf{C},\alpha,\beta:|\mathbf{C}|(x,y)]\models\ \alpha\# I_x=\beta\# I_x\ \Rightarrow\ \alpha=\beta\ .$$

A $\mathcal{G}$-<u>category</u> (or just <u>category</u>, when the meaning is clear) $\mathbf{C}$ is a proto-category such that "Yoneda" is an isomorphism between the $\mathcal{G}$-groupoid $|\mathbf{C}|$ and the underlying groupoid of $\mathbf{C}$, i.e.

$$[x,y:\mathbf{C},f:\mathbf{C}(x,y)]\models \mathrm{Iso}(f)\Rightarrow\exists!_{\alpha:|\mathbf{C}|(x,y)}\ \alpha\# I_x=f\ .$$

Functors between pre-categories and categories are defined just as for proto-categories. We can also define $\mathcal{G}$-<u>proto</u>-<u>groupoids</u>, $\mathcal{G}$-<u>pre</u>-<u>groupoids</u> and $\mathcal{G}$-<u>groupoids</u> : they are $\mathcal{G}$-(etc) such that

$$[x,y:\mathbf{C},f:\mathbf{C}(x,y)]\models \mathrm{Iso}(f)\qquad .$$

Notice that by this definition a $\mathcal{G}$-groupoid is the same (modulo some silly coding) as an object of $\mathcal{G}$, as should be the case, if our terminology is to be consistent.

6.2 Examples

A category object in $\mathcal{D}_1$ is always a $\mathcal{G}$-pre-category but never a $\mathcal{G}$-category, unless it is a discrete category object. Here is an example of a proto-category which is not a pre-category: Take an object $\mathbf{X}$, let $|\mathbf{C}|=\mathbf{X}$, and let $\mathbf{C}(-,-)$ be the identity on $\mathbf{X}\times\mathbf{X}$. That is, $\mathbf{C}$ is the full equivalence relation. "Yoneda" will just be the unique morphism of fibrations $\langle d_0,d_1\rangle\to 1_{\mathbf{X}\times\mathbf{X}}$ in $\mathcal{D}_{\mathbf{X}\times\mathbf{X}}$. Then if $\mathbf{X}$ is not a discrete object "Yoneda" cannot be injective.

Suppose that $\mathcal{G}$ is $\mathrm{Fib}_G/\mathbb{C}$ for a small category $\mathbb{C}$. Let $E:\mathbb{E}\to\mathbb{C}$ be an ordinary Grothendieck fibration (not necessarily of groupoids.) Let $|\mathbb{E}|\subset\mathbb{E}$ be the subcategory with the same objects as $\mathbb{E}$, but where the morphisms are he cartesian arrows of $\mathbb{E}$. This gives an object $|\mathbf{E}|:|\mathbb{E}|\to\mathbb{C}$ of $\mathcal{G}$. Let $\mathbb{E}^*\subset\mathbb{E}^{\to}$ be the full subcategory whose objects are morphisms of $\mathbb{E}$ that are above identity. There is an obvious functor $\langle d_0,d_1\rangle:\mathbb{E}^*\to|\mathbb{E}|\times_{\mathbb{C}}|\mathbb{E}|$ to the pullback of $|\mathbb{E}|$ by itself. It is obvious that this functor is a discrete fibration; in other words we have created a discrete fibration $\mathbf{E}(-,-):\mathbf{E}^{\to}\to|\mathbf{E}|\times|\mathbf{E}|$ in $\mathcal{G}$. The composition and identity operations on $\mathbb{E}$ translate easily to a composition and identity as defined above, that satisfy axiom d), and axiom e) is trivial in this case, because "Yoneda" is the injection $|\mathbb{E}|^*\to\mathbb{E}^*$, where $|\mathbb{E}|^*$ is defined just like $\mathbb{E}^*$. Thus $\mathbf{E}$ is a pre-category. It is not hard to prove that it is a $\mathcal{G}$-category. All the constructions above are obviously functorial, and we get a functor $\mathrm{Fib}/\mathbb{C}\to\mathrm{Cat}(\mathcal{G})$. The reader [well eventually I will do it] can prove that this functor is actually a 2-equivalence; as a hint we will describe the inverse. If $\mathbf{D}$ is a $\mathcal{G}$-category we know that $|\mathbf{D}|:\mathbb{D}\to\mathbb{C}$ is a fibration of groupoids and $\mathrm{Hom}_{\mathbf{D}}(-,-)=\langle d_0,d_1\rangle:\mathbb{H}\to\mathbb{D}\times_{\mathbb{C}}\mathbb{D}$ a discrete fibration. Define a Grothendieck fibration $G:\mathbb{G}\to\mathbb{C}$ as follows. $\mathbb{G}$ has the same objects as $\mathbb{D}$. Given $A,B\in\mathbb{G}$ a morphism of $\mathbb{G}$ is an equivalence class in the set of all pairs (f,α) such that $f\in\mathbb{H}$, $s\in\mathbb{D}$ and $d_1f=\mathrm{dom}(\alpha)$, modulo the relation

$$(f,s)\sim(g,t)\quad\text{iff}\quad |\mathbf{D}|s=|\mathbf{D}|t\ \text{ and }\ (\alpha\#1)\diamond f=g\ ,$$

where α is the unique iso above identity such that $t\alpha=s$. This equivalence allows us identify $\mathcal{G}$-categories with Grothendieck fibrations above $\mathbb{C}$; thus we recover the ordinary theory of fibrations in our world, with the advantage (among many others) that, given a $\mathcal{G}$-category $\mathbf{C}$ we can construct its opposite $\mathbf{C}^{op}$ through a trivial manipulation, which is to be compared with the case of ordinary fibrations.

Everything we have said also applies if $\mathcal{G}$ is $\mathrm{St}_j(\mathrm{Fib}_G/\mathbb{C})$ for a j-topology; that is, a $\mathcal{G}$-category then is "the same" as a stack. [more on this one day]

We have a diagram of inclusions

$$\begin{array}{ccc}
\mathcal{G} = \mathit{Gpd}(\mathcal{G}) & \longrightarrow & \mathit{Cat}(\mathcal{G}) \\
\downarrow & & \downarrow \\
\mathit{PreGpd}(\mathcal{G}) & \longrightarrow & \mathit{PreCat}(\mathcal{G}) \\
\downarrow & & \downarrow \\
\mathit{ProtoGpd}(\mathcal{G}) & \longrightarrow & \mathit{ProtoGpd}(\mathcal{G}) \\
\uparrow & & \uparrow \\
\mathit{Gpd}(\mathfrak{D}_1) & \longrightarrow & \mathit{Cat}(\mathfrak{D}_1)
\end{array}$$

Proposition

The horizontal arrows have a right adjoint. WWW

Here is another important example of a protocategory: let $U(-){:}\mathbf{U}\to\mathbf{X}$ be a discrete fibration. Let us notate the covariant internal action of $\mathbf{U}$ by $*$:

$$[x,y{:}\mathbf{X},\alpha{:}\mathbf{X}(x,y),a{:}U(x)]\ \alpha^*(a){:}U(y)$$

We can construct the "full-sub-proto-category $\mathbf{S}$ generated by U(-) " as follows: take $|\mathbf{S}| = \mathbf{X}$ and given $x,y{:}\mathbf{X}$ let $\mathbf{S}(x,y) = U(x)\Rightarrow U(y)$, which is obviously discrete. Composition in the protocategory is ordinary functional composition; thus "Yoneda" is:

$$[x,y{:}\mathbf{S},\alpha{:}\mathbf{S}(x,y)]\ \lambda_{a:U(x)}\,\alpha^*(a){:}U(x)\Rightarrow U(y) \quad .$$

Definition

A <u>gauge</u> is a discrete fibration $U(-){:}\mathbf{U}\to\mathbf{X}$ such that the construction above yields a category. In other words, a gauge is a discrete fibration with the property that

$$[x,y{:}\mathbf{X},f{:}U(x)\Rightarrow U(y)] \vDash Bi(f) \Rightarrow \exists!_{\alpha:\mathbf{X}(x,y)}\,\forall_{a:U(x)}\,f(a) = \alpha^*(a) \ ,$$

where $Bi(f)$ is the predicate that asserts " f is a bijection". A <u>pointed gauge</u> is a gauge such that there exists $t{:}\mathbf{1}\to\mathbf{X}$ such that the pullback $U(t)$ is identity.

The notion of (mostly pointed) gauge will be used as an abstraction for the category of small sets in a universe. In other words, everything in category theory that pertains to smallness can be defined relative to a gauge.

Examples

Let $(\mathbb{C},\mathfrak{F})$ be a an ordinary display category, where $\mathbb{C}$ is small. Then we can construct a gauge $U(-){:}\mathbf{U}\to\mathbf{X}$ in $Fib_G/\mathbb{C}$: take for $\mathbf{X}$

the fibration $\mathfrak{F}_{pb} \to \mathbb{C}$, where $\mathfrak{F}_{pb}$ is the full category of $\mathbb{C}^{\to}$ whose objects are arrows of $\mathfrak{F}$ and whose morphisms are pullback squares. We want to define a discrete fibration $U: \mathbb{U} \to \mathfrak{F}_{pb}$; given an object $a: A \to I$ in $\mathfrak{F}_{pb}$ the fiber above a is just the set of all splittings of a .

WWW

If $\mathfrak{F}$ contains all isomorphisms, the gauge obtained will be pointed. We see how the concept of gauge is a generalization of the notion of calibration, due to Bénabou [Be1] .

In particular, let $\mathbb{C}$ have finite products. Then we can take for $\mathfrak{F}$ all projections, morphisms of the form $A \times I \to I$. The pointed gauge thus obtained we will call the <u>Lambek</u> gauge. An object of **X** above I is a constant I-indexed family of objects of I . If $\mathbb{C}$ has all finite limits, we can take $\mathfrak{F}$ to be all morphisms of $\mathbb{C}$, and this will give us the <u>Paré–Bénabou</u> gauge. This is the standard way of making $\mathbb{C}$ a small category in $\mathrm{Fib}_G/\mathbb{C}$, as we have said at the beginning of this work.

Proposition

Let $\mathbb{C}$ be a pretopos. Let j be a site structure which is coarser than (contained in) the finite covering topology. Then the Paré–Bénabou gauge is a morphism of j-stacks and is a gauge in $\mathrm{St}_j(\mathrm{Fib}_G/\mathbb{C})$.

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