On a model of choice sequences

Introduction

For application in algebraic geometry, Grothendieck introduced a generalization of the notion of sheaf. The notion of sheaf, introduced previously by Leray and Cartan, was defined over a *topological space* and Grothendieck generalized the notion of topological space to his notion of topos, defined over a *site*. It is quite remarkable that the analysis of the notion of choice sequence by Kreisel and Troesltra [4], involves implicitely a sheaf model also defined over a (non trivial) site [2]. Previously, Beth had (independently of Leray and Cartan) introduced the notion of Beth models, which is a particular case of a sheaf over a topological space, for analysing models of intuitionistic first-order logic.

The goal of this note is to present a variation of the Kreisel-Troelstra model for choice sequences over *Cantor* space and not over Baire space. It was motivated by the work [3], and the point is to give another presentation which exhibits an analogy with the notion of the big Zariski topos in algebraic topology [1].

1 Description of the site and key property

1.1 Big Zariski site

If k is a fixed field, the (big) Zariski site is a site over the opposite of the category of k-algebra (in a fixed universe) with covering of A given by $a_1, \ldots, a_n$ such that 1 belongs to the ideal generated by $a_1, \ldots, a_n$ and localisation maps $A \to A[1/a_i]$. We can have $n = 0$ in which case A is the trivial ring where $1 = 0$.

Any representable presheaf is a sheaf for this topology. The (pre)sheaf represented by an algebra A is the *spectrum* $Sp(A)$ of this algebra. The *affine line* $\mathbb{A}^1$ is the $Sp(k[X])$. Note that $\mathbb{A}^1(A)$ is exactly the carrier set $|A|$ of the algebra A.

1.2 Boolean Zariski site

Here, we take for our base category the opposite of the category of *Boolean algebras* (in a fixed universe). Thus $\mathsf{Hom}(A, B)$ is the set of algebra morphisms $B \to A$.

A *partition of unity* of a Boolean algebra B is given by elements $e_1, \ldots, e_n$ such that $e_i e_j = 0$ if $i \neq j$ and $1 = e_1 + \cdots + e_n$. A covering of a Boolean algebra B is given by a partition of unity $e_1, \ldots, e_n$ of B and the corresponding maps $B \to B[1/e_i]$, where $B[1/e_i]$ is the localisation of B at e_i, which can also be described as the quotient $B/(1 - e_i)$. Note that we can have $n = 0$, in which case B is the *trivial* Boolean algebra.

We write C_n the free Boolean algebra on n generators, where $n = 0, 1, \ldots, \omega$. We write simply C for C_ω which is the Boolean algebra of propositional logic (freely generated by countably many elements). Corresponding to the affine line is the (pre)sheaf $Sp(\mathsf{C}_1)$ represented by the free Boolean algebra C_1 with one generator and $Sp(\mathsf{C}_1)(B) = |B|$ is the carrier set of B.

We have a natural bijection between the sets of morphisms $\mathsf{C} \to B$ and $|B|^{\mathbb{N}}$.

Let $M_{\mathcal{U}}$ be the sheaf model over this site of Boolean algebras in the universe $\mathcal{U}$.

Let X be a set with decidable equality. The sheafification $\overline{X}$ of the constant presheaf X can be described as follow: $\overline{X}(B)$ is the set of formal sums $\Sigma u_i x_i$ where u_i is a partition of unity in B and $\Sigma u_i x_i = \Sigma v_j y_j$ iff $x_i \neq y_j$ (which is decidable) implies $u_i v_j = 0$ in B.

The following result describes exponentiation w.r.t. the sheafification of a constant presheaf.

Lemma 1.1 *If F is a sheaf then $(F^{\overline{X}})(B)$ is canonically isomorphic to $F(B)^X$.*

The base category has a product operation, which in term of Boolean algebra is best thought as a kind of tensor product $B \otimes A$. If A has decidable equality we can choose the carrier set of $B \otimes A$ to be $\overline{|A|}(B)$.

Note that $B \otimes A$, product for the *opposite* category of Boolean algebras, should not be confused with $B \times A$, product of B and A for the category of Boolean algebras.

Note that $B \otimes \mathsf{C}_0$ is canonically isomorphic to B and $B \otimes \mathsf{C}_1$ is canonically isomorphic to $B \times B$.

We can consider Boolean algebras *in the model $M_{\mathcal{U}}$*. As for any algebraic structure, a Boolean algebra in this model is given by a family $F(B)$ of Boolean algebra such that the transition maps $F(B) \to F(B'),\ u \mapsto uf$ for f in $\mathsf{Hom}(B', B)$ are morphisms of Boolean algebras. A morphism $F \to G$ is given by a natural transformation $F(B) \to G(B)$ which is pointwise a morphism of Boolean algebras.
In particular, we can associate to any Boolean algebra A the sheaf $\overline{A}(B) = B \otimes A$ which is the sheafification of the constant presheaf A, if A has decidable equality. (Maybe it also holds in general).

The following well-known basic Lemma plays a fundamental rôle in what follows.
We write π_1 (resp. π_2) in $\mathsf{Hom}(B \otimes A, A)$ (resp. $\mathsf{Hom}(B \otimes A, B)$) for the two projections.

Lemma 1.2 *If F is a sheaf, then $F^{Sp(A)}(B) = F(B \otimes A)$ with evaluation map*

$$F^{Sp(A)}(B) \times G(B) \to F(B),\ (u, f) \mapsto u(f, 1_B)$$

and abstraction map

$$\lambda\alpha : G(B) \to F^{Sp(A)}(B),\ u \mapsto \alpha(u\pi_2, \pi_1)$$

given $\alpha : G \times Sp(A) \to F$

Theorem 1.3 *In $M_{\mathcal{U}}$, we have $Sp(\mathsf{C}_1) = 1 + 1 = 2$ and $Sp(\mathsf{C})$ is Cantor space (the set of functions $2^{\mathbb{N}}$) in the model $M_{\mathcal{U}}$.*

Proof. The first claim follows from the main Lemma: for any sheaf F we have $F^{Sp(\mathsf{C}_1)}(B) = F(B \otimes \mathsf{C}_1) = F(B \times B)$ which is canonically isomorphic to $F(B) \times F(B)$ since F is a sheaf. The second claim then follows since $(Sp(\mathsf{C}_1)^{\mathbb{N}})(B) = (Sp(\mathsf{C}_1)(B))^{\mathbb{N}}$ is naturally in bijection with the set of algebra maps $\mathsf{C} \to B$. □

Finally, we can compute in the model $M_{\mathcal{U}}$ the Boolean algebra generated by n elements.

Lemma 1.4 *In the model $M_{\mathcal{U}}$ the Boolean algebra generated by n elements is $\overline{\mathsf{C}_n}$ for $n = 0, 1, \dots, \omega$*

Proof. For any Boolean algebra E in $M_{\mathcal{U}}$ the set of algebra maps $\overline{\mathsf{C}_n} \to E$ at level B is in bijection with the set of algebra maps $\mathsf{C}_n \to E(B)$ which is $|E(B)|^n$. □

2 Uniform continuity holds in the sheaf model

We still use the main Lemma to compute the set of functions $2^{Sp(\mathsf{C})}$ in the model $M_{\mathcal{U}}$. We have $2^{Sp(\mathsf{C})}(B) = 2(B \otimes \mathsf{C})$ and hence $2^{Sp(\mathsf{C})}$ is the carrier set of the Boolean algebra $\overline{\mathsf{C}}$.
On the other hand, $\overline{\mathsf{C}}$ is the Boolean algebra generated by ω elements *in the model $M_{\mathcal{U}}$* by Lemma 1.4.
We define $x \Vdash a$ in $2 = Sp(\mathsf{C}_1)$ for x in $Sp(\mathsf{C})$ and a in $\overline{\mathsf{C}}$ by taking $f \Vdash (\Sigma u_i a_i)$ to be $\Sigma u_i f(a_i)$ where f is a morphism $\mathsf{C} \to B$ and $\Sigma u_i a_i$ is in $B \otimes \mathsf{C}$.
We can summarize the situation as follows, which expresses a perfect duality between $\overline{\mathsf{C}}$ and $Sp(\mathsf{C})$ in the model $M_{\mathcal{U}}$, and in particular implies that any map $Sp(\mathsf{C}) \to 2$ is uniformely continuous.

Theorem 2.1 *Any map $Sp(\mathsf{C}) \to 2$ is of the form $x \mapsto x \Vdash a$ for some (unique) a in $\overline{\mathsf{C}}$ and any algebra map $\overline{\mathsf{C}} \to \overline{\mathsf{C}_0}$ is of the form $a \mapsto x \Vdash a$ for some (unique) x in $Sp(\mathsf{C})$.*

More generally, we have such a perfect duality between $\overline{A}$ and $Sp(A)$ whenever A has a decidable equality (which is the same as being overt). We don't know if the statement holds as well for a general Boolean algebra.

3 How this model is usually presented

The only objects in the base category relevant for this model are finite products of localisations of Cantor space. All these objects are isomorphic to Cantor space. Thus, one can work with the equivalent category of Cantor space and endomorphisms. This is what is done in [4, 2] for Baire space and [3] for Cantor space. An advantage of our formulation is that Theorem 1.3 has a transparent proof.

References

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