AXIOMATIC SET THEORY 4

Complete Boolean Algebra

1. If B is a *complete* Boolean algebra, that is a Boolean algebra such that any subset has a least upper bound (and a greatest lower bound), show that the infinite distributivity law holds
$$a \cdot \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \cdot b_i)$$
(this is a result due to von Neumann).

2. If X is a topological space, we write $\overline{A}$ the closure of a subset $A \subseteq X$ and $int(A)$ the interior of A. An open subset U is *regular* iff $U = int(\overline{U})$. Show that the set of all regular open set is a complete Boolean algebra. Show that $\bigwedge_i U_i$ is $int(\cap_i U_i)$ while $\bigvee_i U_i$ is $int(\overline{\cup_i U_i})$.

3. For any set I if X is the space $\{0,1\}^I$ with the product topology (with $\{0,1\}$ discrete) show that the complete Boolean algebra B of regular open subsets of X has the following property: if $x_j \in B,\ j \in J$ satisfies $x_j \neq 0$ and $x_{j_1} \cdot x_{j_2} = 0$ whenever $j_1 \neq j_2$ then J is at most countable. (Hint: consider the usual product measure on X.)

4. If S is any set, we can consider it to be a topological space with the discrete topology. We let then X be the product space S^ω, and B be the complete Boolean algebra of regular open subsets of X. Let $\Omega_{(n,s)}$ be $\{f \in X \mid f(n) = s\}$ for $n \in \omega,\ s \in S$. Show that $\Omega_{(n,s)}$ is an open closed subset of X, and hence an element of B. Show that we have in B

   - $\Omega_{(n,s_1)} \cdot \Omega_{(n,s_2)} = 0$ if $s_1 \neq s_2$
   - $1 = \bigvee_{s \in S} \Omega_{(n,s)}$ for all $n \in \omega$
   - $1 = \bigvee_{n \in \omega} \Omega_{(n,s)}$ for all $s \in S$

   This Boolean algebra B is called a *collapsing* algebra: in the Boolean-valued model over B the cardinality of S collapses and is countable (this follows from the relations above: if we read $\Omega_{(n,s)}$ as being the truth value of $f(n) = s$, these relations express that f is a surjective map $\omega \to S$.)

   If we assume now S to be an ordinal, it has been noticed by Sololvay that, though the cardinality of B is arbitrary large, B is *always* countably generated by the sets $X_{n,m} = \{f \in X \mid f(n) \leq f(m)\}$. This shows that there is no free complete Boolean algebra over a countable set.