Sheaf models of type theory

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Goal of the talk

Sheaf models of *higher order logic* have been fundamental for establishing *consistency* of logical principles

E.g. consistency of Brouwer's *fan theorem*

Or of the existence of the algebraic closure of a field (Joyal) or of some axioms of non standard analysis (Moerdijk, Palmgren)

This also can be used to establish *independence results*

E.g. independence of the principle of countable choice

Origin: *algebraic topology* (Leray, Cartan) and *logic* (Beth, 1956)

Goal of the talk

Can we extend this notion of sheaf models to *dependent type theory*?

The problem is how to model the *universes*

«The collection of sheaves don't form a sheaf»

If we define $F(V)$ to be the collection of all $\mathcal{U}$-sheaves on V then F is a presheaf which is not a sheaf in general, since glueing will only be defined up to isomorphism

This basic fact was the motivation for the notion of *stacks*

Goal of the talk

We present a possible version of the notion of sheaf model for dependent type theory ("cubical" stacks)

It applies to type theory extended with the *univalence axiom* and higher inductive types

Theorem 1: *The principle of countable choice is independent of type theory with the univalence axiom and propositional truncation*

Theorem 2: *Type theory with the univalence axiom and propositional truncation is compatible with Brouwer's fan theorem*

This generalizes previous works with Bassel Mannaa and Fabian Ruch on the groupoid model, and is the result of several discussions with Christian Sattler

Countable choice

$\Pi(A : \mathsf{N} \to \mathsf{U})\ (\Pi(n : \mathsf{N})\ \|A\ n\|) \to \|\Pi(n : N) A\ n\|$

In this statement $\|T\|$ denotes the *propositional truncation* of T

We are going to build a model with a particular family A where

-the hypothesis $\Pi(n : \mathsf{N})\ \|A\ n\|$ holds

-the conclusion $\|\Pi(n : N) A\ n\|$ does not hold

Presheaf model of type theory

We work in a (constructive) set theory with universes $\mathcal{U}_0 \in \mathcal{U}_1 \in \cdots \subseteq \mathcal{U}_\omega$

We have a base category $\mathcal{C}$ in $\mathcal{U}_0$

We write $I, J, K, \dots$ the objects of $\mathcal{C}$

$Yo(I)$ denotes the presheaf represented by I

Define the set of contexts $\Gamma, \Delta, \dots$ to be the set of $\mathcal{U}_\omega$-presheaves on $\mathcal{C}$

$\mathsf{Type}_n(\Gamma)$ set of $\mathcal{U}_n$-presheaves on the category of elements of Γ

$\mathsf{Elem}(\Gamma, A)$ set of global sections of $A \in \mathsf{Type}_n(\Gamma)$

Presheaf model of type theory

Composition gives a substitution operation $A\sigma$ in $\mathsf{Type}_n(\Delta)$ if $\sigma : \Delta \to \Gamma$

Similarly, we define $a\sigma$ in $\mathsf{Elem}(\Delta, A\sigma)$ if a is in $\mathsf{Elem}(\Gamma, A)$ and $\sigma : \Delta \to \Gamma$

We have a canonical context extension operation $\Gamma.A$ for A in $\mathsf{Type}_n(\Gamma)$

$p : \Gamma.A \to \Gamma$ and q in $\mathsf{Elem}(\Gamma.A, Ap)$

Any $\mathcal{U}_n$-presheaf F defines a constant family $\overline{F} \in \mathsf{Type}_n(\Gamma)$

Presheaf model of type theory

We have a natural *product* operation

$\Pi(A, B) \in \mathsf{Type}_n(\Gamma)$ if $A \in \mathsf{Type}(\Gamma)$ and $B \in \mathsf{Type}(\Gamma.A)$

Furthermore $\Pi(A, B)\sigma = \Pi(A\sigma, B(\sigma p, q))$

We also have

an *abstraction* operation $\lambda\ b \in \mathsf{Elem}(\Gamma, \Pi(A, B))$ for $b \in \mathsf{Elem}(\Gamma.A, B)$

and an *application* operation $\mathsf{app}(c, a)$ in $\mathsf{Elem}(\Gamma, B[a])$ whenever c is in $\mathsf{Elem}(\Gamma, \Pi(A, B))$ and a in $\mathsf{Elem}(\Gamma, A)$

satisfying the required equations

Presheaf model of type theory: universes

Type_n with substitution defines a presheaf on the category of contexts

It is *continuous* and hence *representable* by $U_n(I) = \mathsf{Type}_n(Yo(I))$

We have natural bijections $\mathsf{Type}_n(\Gamma) \simeq \Gamma \to U_n \simeq \mathsf{Elem}(\Gamma, \overline{U_n})$

Base category

There are several possible choices for the base category

We can take the *Lawvere category* associated to the *equational theory* of *bounded distributive lattices* or de Morgan algebra, or Boolean algebra

The morphisms of the base category can be thought of as *substitutions*

We can also take the category of nonempty finite sets and arbitrary maps

In this case, we get *symmetric* simplicial sets

Base category

What matters is that we have a *segment* i.e. a presheaf $\mathbb{I}$ with two distinct elements 0 and 1 satisfying

(1) $\mathbb{I}$ has a connection structure, i.e. maps $(\wedge), (\vee) : \mathbb{I} \to \times \to \mathbb{I}$ satisfying $x \wedge 1 = x = 1 \wedge x,\ x \wedge 0 = 0 = 0 \wedge x$ and $x \vee 1 = 1 = 1 \vee x,\ x \vee 0 = x = 0 \vee x$ and

(2) We have a functor J^+ on $\mathcal{C}$ with a natural isomorphism $Yo(J^+) \simeq Yo(J) \times \mathbb{I}$

We get a notion of path by exponentiation to this interval $\mathbb{I}$

In order to define a notion of ⟪open boxes⟫, we need a further notion of *cofibrations*

Complete Cisinski model structures

We can always take as *cofibrations* the monomorphisms $m : A \to B$ such that each $m_I : A(I) \to B(I)$ has *decidable* image

This corresponds to the choice $\mathbb{F}(J) =$ decidable sieves on J

Classically this is the same as taking *all* monomorphims as cofibrations

Given a notion of segment Cisinski has shown how to define a *model structure* where cofibrations are monomorphisms

This does not use the hypotheses (1) and (2) on the segment

Complete Cisinski model structures

Christian Sattler has also shown (using what I will present next) how to define another model structure, under the hypotheses (1) and (2), which has the same notion of *fibrant objects* and (classically) *cofibrations* as the one of Cisinski model structure

It follows that it *coincides* with Cisinski model structure and provides a proof that this class of Cisinski model structures are *complete*

In general the notion of cofibration will be given by a subpresheaf of Ω

QUESTION: *do some of these model structures represent the standard homotopy theory of CW complexes?*

«Inner» models

Using the segment $\mathbb{I}$ and a notion of cofibrations, we can define a set of «filling structures» $\mathsf{Fill}(\Gamma, A)$

An element of $\mathsf{Fill}(\Gamma, A)$ represents a generalized «path lifting» operation for the projection $p : \Gamma.A \to \Gamma$

It expresses that the type of all path liftings is contractible (for a given path in the base and starting point)

It can also be seen as a (generalized) open box filling operation

$p_A : \Gamma.A \to \Gamma$ *is a (naive) fibration if, and only if* A *has a filling structure*

⟪Inner⟫ models

Define $\mathsf{Fib}_n(\Gamma)$ in $\mathcal{U}_{n+1}$

$\mathsf{Fib}_n(\Gamma)$ set of pairs (X, c) with $X \in \mathsf{Type}_n(\Gamma)$ and $c \in \mathsf{Fill}(\Gamma, X)$

$\mathsf{Elem}_\mathsf{F}(\Gamma, (X, c)) = \mathsf{Elem}(\Gamma, X)$

We get a new ⟪proof relevant⟫ inner model of the presheaf model

⟪Inner⟫ models

We can lift the product operation at this level

$\pi(c_A, c_B) \in \mathsf{Fill}(\Gamma, \Pi(A, B))$ if $c_A \in \mathsf{Fill}(\Gamma, A)$ and $c_B \in \mathsf{Fill}(\Gamma.A, B)$

Furthermore $\pi(c_A, c_B)\sigma = \pi(c_A\sigma, c_B(\sigma p, q))$

We can define a product operation for this new model

$\Pi((A, c_A), (B, c_B)) = (\Pi(A, B), \pi(c_A, c_B))$

We don't need to change the abstraction and application operations

$\mathsf{Elem}_{\mathsf{F}}(\Gamma, (X, c)) = \mathsf{Elem}(\Gamma, X) \qquad \Gamma.(X, c) = \Gamma.X$

⟪Inner⟫ models

What about universes?

Fib_n *is* continuous *and hence* representable *by* $F_n(I) = \mathsf{Fib}_n(Yo(I))$

We have a natural isomorphism $\Gamma \to F_n \simeq \mathsf{Fib}_n(\Gamma)$

We can then build c_n in $\mathsf{Fill}(\Gamma, \overline{F_n})$

In this way we define $\mathsf{U}_n = (\overline{F_n}, c_n)$ in $\mathsf{Fib}_{n+1}(\Gamma)$

Theorem: *We get a model of type theory with the univalence axiom and higher inductive types*

⟪Inner⟫ models

The definition of the set $\mathsf{Fill}(\Gamma, A)$ depends on the interval and of the notion of cofibrations which can be seen as a subpresheaf $\mathbb{F}$ of Ω

If we take for $\mathbb{F}(I)$ all decidable sieves on I we get classically all monomorphisms

Differences with the simplicial set model

A type in the new model is a presheaf *together with* a Kan operation

For this model $\mathsf{Fib}(\Gamma, A)$ is not a subset of $\mathsf{Type}(\Gamma, A)$

For the simplicial set model

-to be a Kan fibration is a *property* and not a *structure*

-axiom of choice seems needed to prove that the universe of Kan types is Kan (at least all known arguments so far use choices)

Differences with the simplicial set model

An element of Fill(Γ, A) can be thought of as an *explicit filling operation*

It fills a given open box in A over a filled box in Γ

If A, B are in Type(Γ) with given filling operations there is thus a notion of *structure preserving* maps $w : A \to B$ which is a *property* of such a map

We are going next to use the Kan structure to define a new notion of stacks

Presheaf extension of the cubical set model

The cubical set model generalizes automatically to any presheaf extensions

Given another category $\mathcal{D}$ in $\mathcal{U}_0$ with objects $X, V, L, \ldots$ we now define a context as being a $\mathcal{U}_\omega$-presheaf on $\mathcal{D} \times \mathcal{C}$

A context Γ is given by a family of sets $\Gamma(X|I)$ in $\mathcal{U}_\omega$ with restriction maps

Given X we can consider the cubical set $\Gamma(X) : I \mapsto \Gamma(X|I)$

Presheaf extension of the cubical set model

$\mathbb{I}_{\mathcal{D}}(X|J) = \mathbb{I}(J)$ defines a segment

There are several choices for the cofibrations $\mathbb{F}_{\mathcal{D}}$

We can take

CHOICE 1: $\mathbb{F}_{\mathcal{D}}(X|J) = \mathbb{F}(J)$

CHOICE 2: $\mathbb{F}_{\mathcal{D}}(X|J)$ all decidable sieves on $X|J$

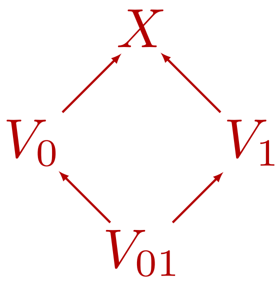
Two basic examples

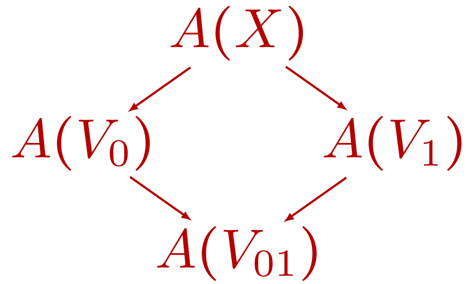
Sierpinski's space

$$\begin{array}{ccc} X & \qquad\qquad & A(X) \\ \uparrow & & \downarrow \\ V & & A(V) \end{array}$$

To analyse the notion of presheaf

Two basic examples





To analyse the notion of sheaf

Example 0

Sierpinski's space

$$\begin{array}{ccc} X & \qquad\qquad & A(X) \\ \uparrow & & \downarrow \\ V & & A(V) \end{array}$$

A diagram and an associated "cubical presheaf"

Example 0

For CHOICE 1, a Kan structure for A consists in

-a Kan structure for each cubical set $A(X), A(V)$

-the property that each restriction maps are *structure preserving*

Note that, for CHOICE 1, the map $A(X) \to A(V)$ does not need to be a *fibration* (i.e. may not have a fibration structure)

Presheafs with such Kan structures still form a model of type theory with univalence and higher inductive types

Example 0

For CHOICE 2 we add the new open box of $(X, J) \times \mathbb{I}$

$(V, J) \times \mathbb{I} \cup (X, J|\psi) \times \mathbb{I} \cup (X, J) \times 0$

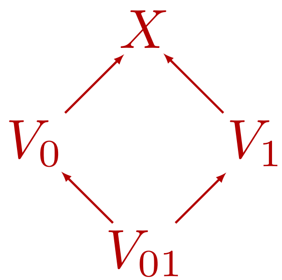
and this implies that $A(X) \to A(V)$ is a *fibration*

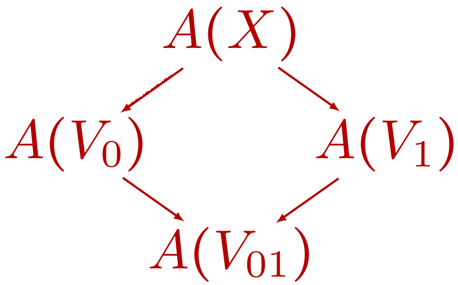
Example 0

The nerve of any groupoid has a filling structure for CHOICE 1

The nerve of this particular groupoid has no filling structure for CHOICE 2

Example 1





A diagram and an associated "cubical presheaf"

Restriction maps are cubical set maps; it is natural to write these maps as $u \mapsto u|V_0,\ A(X) \to A(V_0)$ with $(u|V_0)|V_{01} = (u|V_1)|V_{01} = u|V_{01}$

Example 1

In both CHOICES 1 and 2, A Kan structure for A will define

-a Kan structure for each cubical set $A(X), A(V_0), A(V_1), A(V_{01})$

-the property that each restriction maps are *structure preserving*

with for CHOICE 2, some extra conditions: the square has to be *reedy fibrant*

Presheafs with such Kan structures still form a model of type theory with univalence and inductive types

Descent data

We now want to express that X is covered by V_0 and V_1

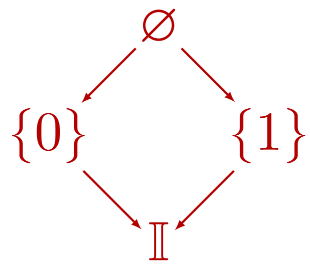
We first define the presheaf of *descent data* $D(A)$

An element of $D(A)(X|I)$ is of the form (u_0, u_1, u_{01}) with $u_0 \in A(V_0|I)$ and $u_1 \in A(V_1|I)$ and u_{01} a path $u_0|V_{01} \rightarrow u_1|V_{01}$

Note that we only require a *path* between u_0 and u_1 and *not* a strict equality

Descent data and stacks

Christian Sattler noticed that we have a canonical *isomorphism* $D(A) \simeq A^F$ where F is the cubical presheaf



This provides a simple proof that D lifts at the level of Kan structure

Descent data and stacks

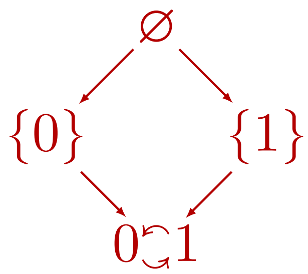
We have a canonical map $m_A : A \to D(A)$

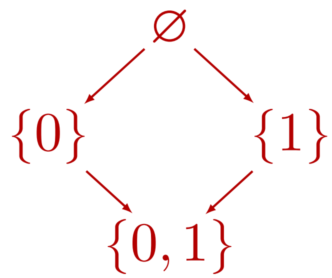
A *stack* structure for A is then an equivalence structure for this map

Remark: D defines a (strict) monad, which is *idempotent* in the sense that $m_{D(A)}$ and $D(m_A)$ are *path equal*

This is a (new) example of a *left exact modality* as studied by Egbert Rijke, Mike Shulman and Bas Spitters

Descent data and stacks





The first example is not a stack, the second example is a stack (a set)

Descent data and stacks

The notion of stack structure is internally defined $S(A) = \mathsf{isEquiv}\ m_A$

A stack structure is an element of $\mathsf{Elem}_{\mathsf{F}}(\Gamma, S(A))$

Stack structures lift to dependent products and sums (can be proved internally)

Descent data and stacks

Also we can prove $S(\Sigma(X : \mathsf{U}_n)S(X))$

The proof uses *univalence* in an essential way

We can define a map $L_n : D(\mathsf{U}_n) \to \mathsf{U}_n$ (dependent product) which satisfies

$$L_n(m_{\mathsf{U}_n}(A)) = D(A)$$

This implies that L_n is a *left inverse* of m on types that have a stack structure, since then $D(A)$ and A are equal by univalence, and hence that L_n is homotopy inverse of m since D is idempotent

Stack model

Define $\mathsf{Stack}_n(\Gamma)$ to be the set of pairs (A, s)

$A \in \mathsf{Fib}(\Gamma)$ and $s \in \mathsf{Elem}_\mathsf{F}(\Gamma, S(A))$

Define $\mathsf{Elem}_\mathsf{S}(\Gamma, (A, s))$ to be $\mathsf{Elem}_\mathsf{F}(\Gamma, A)$

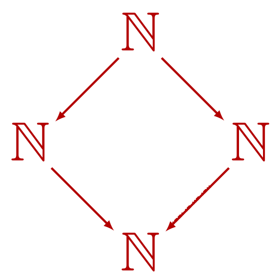
We get in this way a model of type theory

This model still satisfies univalence and interprets higher inductive types

This works both for CHOICES 1 and 2

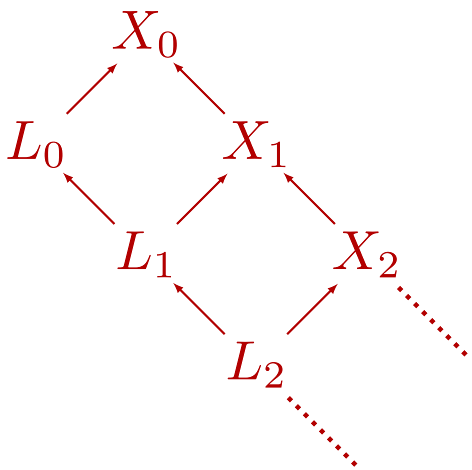
Stack model

The following constant presheaf is a stack



Example 2: Countable choice

We now consider the following space, where X_n is covered by L_n and X_{n+1}



Example 2: Countable choice

We now have a *family* of idempotent monads, indexed by the coverings

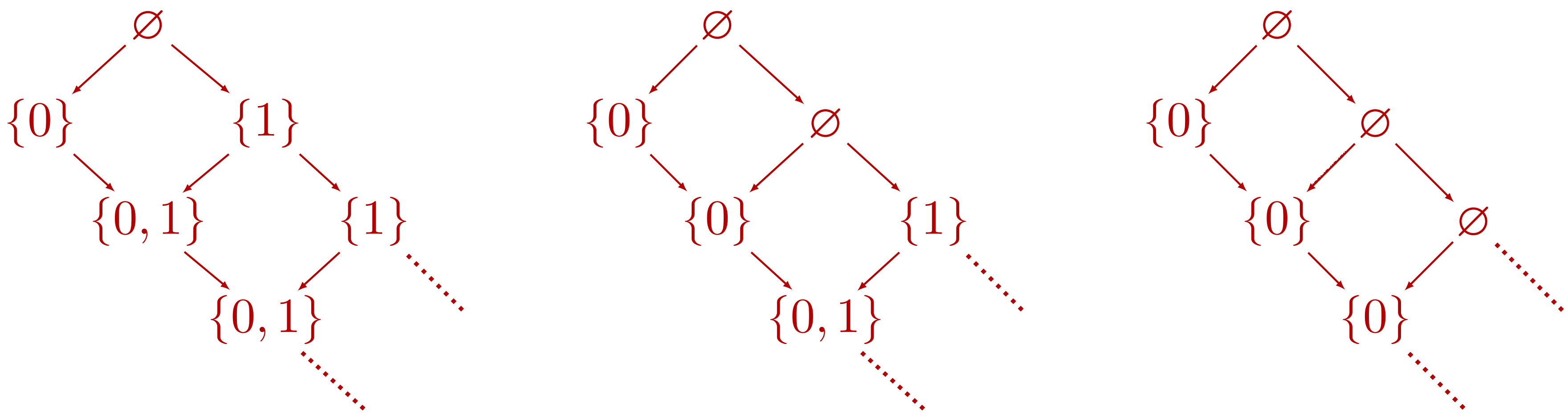
We have a presheaf Cov on the given space X

We have a family D_c of idempotent monads over Cov

The notion of stacks generalize and form a model of type theory

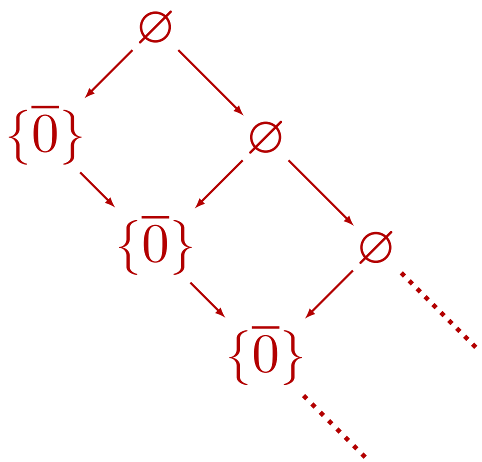
Example 2: Countable choice

We then can define a family of sets (stacks) $A\ n$, e.g. for $A\ 0,\ A\ 1$ and $A\ 2$



Example 2: Countable choice

$\Pi(n : N) A\ n$ is (a proposition) is *not* globally inhabited and $\|A\ n\|$ *is* globally inhabited *because* of the stack condition



Example 2: Countable choice

This model provides thus an explicit counter-example to countable choice

Note the use of a non well-founded diagram

Not clear how this can be adapted (classically) to the setting of simplicial sets

Example 3: Markov principle

Let C be the Boolean algebra corresponding to Cantor space

The base category is the poset of nonzero elements of C

A covering is a partition of unity. Note that all covering are disjoint (no compatibility conditions), and that $D_c(A) \simeq A^{F_c}$ where each F_c is a subsingleton of the presheaf model

Theorem: *Markov's principle does not hold in the corresponding stack model of type theory. Actually, its* negation *holds (Bassel Mannaa).*

Corollary: *Markov's principle cannot be proved in type theory with univalence*

Example 4: Fan theorem

Let $\mathcal{D}^{op}$ be a full subcategory of the category of Boolean algebra having for objects localizations of finite power of C

A covering of an object is given by a partition of unity and corresponding localizations (Zariski topology)

Lemma: $2(B|J) = B$ *and* 2^N *is represented by* $(C, \emptyset)$

Theorem: *Brouwer's fan theorem holds in the corresponding stack model of type theory*