

Mathematical Logic Exercises 1

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Section 1.1

Question 3: Show that the relation “is a subformula of” is transitive.

The definition of $Sub(\varphi)$:

$$\begin{aligned}Sub(\varphi) &= \{\varphi\} \text{ for atomic } \varphi. \\Sub(\varphi_1 \Box \varphi_2) &= Sub(\varphi_1) \cup Sub(\varphi_2) \cup \{\varphi_1 \Box \varphi_2\} \\Sub(\neg \varphi) &= Sub(\varphi) \cup \{\neg \varphi\}\end{aligned}$$

(p12) Formula ψ is a subformula of φ if $\psi \in Sub(\varphi)$.

We want to show that if (1) φ_1 is a subformula of φ_2 , and (2) φ_2 is a subformula of φ_3 , then (3) φ_1 is a subformula of φ_3 .

Idea: reduce it to the transitivity of \subseteq .

Lemma: If $\psi \in Sub(\varphi)$, then $Sub(\psi) \subseteq Sub(\varphi)$.

Proceed by induction over φ :

φ **atomic (base case):** By definition, $Sub(\varphi) = \{\varphi\}$ since φ is atomic. So, if $\psi \in Sub(\varphi)$, then $\psi = \varphi$, and hence $Sub(\psi) \subseteq Sub(\varphi)$.

$\varphi = \neg \varphi_1$ **(inductive step):** $Sub(\varphi) = Sub(\neg \varphi_1)$, which by definition is equal to $Sub(\varphi_1) \cup \{\varphi\}$. So, if $\psi \in Sub(\varphi)$ we know that $\psi \in Sub(\varphi_1)$ or $\psi \in \{\varphi\}$. We treat each case separately:

1. $\psi \in Sub(\varphi_1)$: By the induction hypothesis, it follows that $Sub(\psi) \subseteq Sub(\varphi_1)$. But $Sub(\varphi_1) \subseteq Sub(\varphi_1) \cup \{\varphi\} = Sub(\varphi)$. So, $Sub(\psi) \subseteq Sub(\varphi)$.
2. $\psi \in \{\varphi\}$: From $\psi \in \{\varphi\}$ we know that $\psi = \varphi$, and therefore $Sub(\psi) \subseteq Sub(\varphi)$ (cf. base case).

In both cases we have proven $Sub(\psi) \subseteq Sub(\varphi)$, hence the inductive step is done.

$\varphi = \varphi_1 \Box \varphi_2$ **(inductive step):** $Sub(\varphi) = Sub(\varphi_1 \Box \varphi_2)$, which by definition is equal to $Sub(\varphi_1) \cup Sub(\varphi_2) \cup \{\varphi\}$. So from $\psi \in Sub(\varphi)$ we can distinguish three cases: $\psi \in Sub(\varphi_1)$, or $\psi \in Sub(\varphi_2)$, or $\psi \in \{\varphi\}$. We treat each case separately:

1. $\psi \in Sub(\varphi_1)$: By the induction hypothesis for φ_1 , it follows that $Sub(\psi) \subseteq Sub(\varphi_1)$. In turn, $Sub(\varphi_1) \subseteq Sub(\varphi_1) \cup Sub(\varphi_2) \cup \{\varphi\} = Sub(\varphi)$, hence $Sub(\psi) \subseteq Sub(\varphi)$.
2. $\psi \in Sub(\varphi_2)$: By the induction hypothesis for φ_2 , it follows that $Sub(\psi) \subseteq Sub(\varphi_2)$. In turn, $Sub(\varphi_2) \subseteq Sub(\varphi_1) \cup Sub(\varphi_2) \cup \{\varphi\} = Sub(\varphi)$, hence $Sub(\psi) \subseteq Sub(\varphi)$.
3. $\psi \in \{\varphi\}$: From $\psi \in \{\varphi\}$ we know that $\psi = \varphi$, and therefore $Sub(\psi) \subseteq Sub(\varphi)$ (cf. base case).

In all three cases we have proven $Sub(\psi) \subseteq Sub(\varphi)$, hence the inductive step is done.

Transitivity of “is a subformula of” follows from the transitivity of \subseteq , and the reflexivity of $Sub(\varphi)$. (The latter is obvious from the definition.)

Question 9: Show that a proposition with n connectives has at most $2n + 1$ subformulas.

We want to show that if φ has n connectives, then $|Sub(\varphi)| \leq 2n + 1$.

Proceed by induction on the number of connectives n :

Base case: By the definition of $Sub(\varphi)$, if φ is atomic ($n = 0$), then it has 1 subformula.

Inductive Hypothesis: If φ has n connectives, then $|Sub(\varphi)| \leq 2n + 1$.

Inductive Argument: Assume φ has n connectives. There are two cases:

1. φ is $\neg\varphi_1$, so φ_1 has $n - 1$ connectives. By the inductive hypothesis, $|Sub(\varphi_1)| \leq 2n - 1$ and by the definition of $Sub(\varphi)$, $|Sub(\varphi)| = |Sub(\varphi_1) \cup \{\neg\varphi_1\}| \leq (2n - 1) + 1 = 2n \leq 2n + 1$.
2. φ is $\varphi_1 \Box \varphi_2$. Let i, j be the number of connectives in φ_1, φ_2 respectively. We know $n = i + j + 1$, and specifically $i < n$ and $j < n$. By the inductive hypothesis we have $|Sub(\varphi_1)| \leq 2i + 1$, $|Sub(\varphi_2)| \leq 2j + 1$ and so by the definition of $Sub(\varphi)$, $|Sub(\varphi)| = |Sub(\varphi_1) \cup Sub(\varphi_2) \cup \{\varphi_1 \Box \varphi_2\}| \leq |Sub(\varphi_1)| + |Sub(\varphi_2)| + 1 = (2i + 1) + (2j + 1) + 1 = 2n + 1$.

Alternatively, induction over ranks, etc. can be used. The proof is structurally the same.

Section 1.2

Question 1b : Determine (using a truth table) if $\varphi \rightarrow ((\psi \rightarrow \sigma) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma)))$ is a tautology.

Let f denote this formula.

φ	ψ	σ	f
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

So f is a tautology.

Question 2: Prove the following properties.

We need the following definitions:

Definition 1.2.1 of valuations, specifically:

$$v(\varphi \rightarrow \psi) = 0 \Leftrightarrow v(\varphi) = 1 \text{ and } v(\psi) = 0.$$

Definition 1.2.4: $\varphi \models \psi$ holds if, for all v such that $\llbracket \varphi \rrbracket_v = 1$, then $\llbracket \psi \rrbracket_v = 1$.

(a) $\varphi \models \varphi$.

Follows directly from the definition of \models .

(b) Show $\varphi \models \psi$ and $\psi \models \sigma \Rightarrow \varphi \models \sigma$.

Let v satisfy φ , i.e. $\llbracket \varphi \rrbracket_v = 1$, then $\llbracket \psi \rrbracket_v = 1$ because $\varphi \models \psi$. Similarly, $\llbracket \sigma \rrbracket_v = 1$, and therefore, by the definition of \models , $\varphi \models \sigma$.

(c) $\models \varphi \rightarrow \psi \Leftrightarrow \varphi \models \psi$.

(\Rightarrow) Let v be any valuation¹ for $\varphi \rightarrow \psi$. Proceed by cases:

(a) If $\llbracket \varphi \rrbracket_v = 0$, then the antecedent in Definition 1.2.4 does not hold, and so, in this case, $\varphi \models \psi$ vacuously.

(b) If $\llbracket \varphi \rrbracket_v = 1$, then by Definition 1.2.1, $\llbracket \psi \rrbracket_v = 1$.

Thus $\varphi \models \psi$ by Definition 1.2.4.

(\Leftarrow) Let v be a valuation for φ . Proceed by cases:

(a) If $\llbracket \varphi \rrbracket_v = 0$, then $\varphi \models \psi$ for any truth value of ψ , and moreover $\llbracket \varphi \rightarrow \psi \rrbracket_v = 1$ by Definition 1.2.1.

(b) If $\llbracket \varphi \rrbracket_v = 1$, then by Definition 1.2.4 $\llbracket \psi \rrbracket_v = 1$ as well, so $\llbracket \varphi \rightarrow \psi \rrbracket_v = 1$ by Definition 1.2.1.

Thus $\models \varphi \rightarrow \psi$ by Definition 1.2.4.

¹We say “valuation for ϕ ” rather than just “valuation” because we need to specify what variables will be assigned values. We say “let v be a satisfying valuation for ϕ ” if we need $\llbracket \phi \rrbracket_v = 1$.