Mathematical Logic Exercises 4

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Section 2.2

Question 1: Write down the similarity type for the following structures:

1. $\langle \mathbb{Q}, <, 0 \rangle$
   $\langle 2; ; 1 \rangle$
2. $\langle \mathbb{N}, +, \cdot, S, 0, 1, 2, 3, 4, \ldots, n, \ldots \rangle$, where $S(x) = x + 1$.
   $\langle ; 2, 2, 1; \aleph_0 \rangle$ (where $\aleph_0$ is the cardinality of the set of natural numbers)
3. $\langle \mathcal{P}(\mathbb{N}), \subseteq, \cup, \cap, {}^c, \emptyset \rangle$
   $\langle 2; 2, 2, 1; 1 \rangle$

Section 2.3

Question 1: Write down an alphabet for the languages of the types given in Exercise 1 of Section 2.2.

All languages have the core first-order logic objects:

- Predicate symbols: $\dot{=}$
- Variables: $x_0, x_1, x_2, \ldots$ (countably many)
- Connectives: $\vee, \wedge, \rightarrow, \neg, \leftrightarrow, \bot, \forall, \exists$
- Auxiliary symbols: (,),.

The following list what is added to this core.

(i) $\langle \mathbb{Q}, <, 0 \rangle$

1. Predicate symbols: LT (arity 2)
2. Function symbols: (none)
3. Constant symbols: $\{\overline{0}\}$

(ii) $\langle \mathbb{N}, +, \cdot, S, 0, 1, 2, 3, 4, \ldots, n, \ldots \rangle$, where $S(x) = x + 1$.

1. Predicate symbols: (none)
2. Function symbols: $Plus, Mult$ (arity 2), $Succ$ (arity 1)
3. Constant symbols: $\{\overline{0}, \overline{1}, \ldots\}$

(iii) $\langle \mathcal{P}(\mathbb{N}), \subseteq, \cup, \cap, {}^c, \emptyset \rangle$

1. Predicate symbols: $SubsetEq$ (arity 2)
2. Function symbols: $Union, Intersection$ (arity 2), $Complement$ (arity 1)
3. Constant symbols: $\overline{\emptyset}$

(vi) $\langle \mathbb{R}, 1 \rangle$

1. Predicate symbols: (none)
2. Function symbols: (none)
3. Constant symbols: $\{\overline{1}\}$

(vii) $\langle \mathbb{R} \rangle$

1. Predicate symbols: (none)
2. Function symbols: (none)
3. Constant symbols: $\emptyset$

(viii) $\langle \mathbb{R}, \mathbb{N}, <, T, \ ^2, ||, - \rangle$ where $T(a, b, c)$ is the relation "b is between a and c"; ^{2} is the square function, $-$ is the subtraction function and $||$ the absolute value.

1. Predicate symbols: $N, LT, \overline{T}$
2. Function symbols: $Subtract$ (arity 2), $Square, Abs$ (arity 1)
3. Constant symbols: $\emptyset$

The use of $\mathbb{N}$ as a one-place predicate symbol which is true only if the argument denotes a natural number is explained on p78 of the third edition of the textbook (Section 2.5).

Question 2: Write down five terms of the language belonging to Exercise 1 (iii), (viii). Write down two atomic formulas of the language belonging to Exercise 1 (vii) and two closed atoms for Exercise 1 (iii), (vi).

1 (iii) Term examples: $x, \overline{\emptyset}, Union(x, \overline{\emptyset}), Intersection(y, Union(\overline{\emptyset}, Complement(z)))$, ...

1 (viii) Term examples: $x, Square(x), Subtract(x, y), Abs(z), \ldots$

1 (vii): Atomic formula examples: $x \doteq y, y \doteq z$.

All atoms must be equality between two variables. . .

1 (iii): Closed atom examples: $SubsetEq(\overline{\emptyset}, \overline{\emptyset}), Complement(\overline{\emptyset}) \doteq \overline{\emptyset}$, ...

1 (vi): Closed atom examples: $\bot, \overline{1} \doteq \overline{1}$. There are only two (prove it!).

Question 4: Check which terms are free in the following cases, and carry out the substitution:

van Dalen's definition of "t is free for x in φ" is a little ambiguous (specifically (iii)). Here is a less ambiguous one:

Term t is free for variable x in formula φ if:

1. φ is atomic.
2. $\varphi \equiv \neg\varphi_1$ and t is free for x in φ_1.
3. $\varphi \equiv \varphi_1 \square \varphi_2$ and t is free for x in both φ_1 and φ_2.
4. The interesting case: $\varphi \equiv \exists y.\varphi_1$ or $\varphi \equiv \forall y.\varphi_2$ and either:
   - $x = y$, or
   - $x \neq y$ and $y \notin FV(t)$ and t is free for x in φ_1.

Note that, as t is a term, *all* variables in t are free. Thus, it suffices to check if $y \in Vars(t)$.

(d) $\overline{0} + y$ for y in $\exists x(y \doteq x)$.

$\overline{0} + y$ is free for y in $\exists x(y \doteq x)$, and the resulting term is $\exists x((\overline{0} + y) \doteq x)$.

(f) $x + w$ for z in $\forall w(x + z \doteq \overline{0})$.

$x + w$ is not free for z in $\forall w(x + z \doteq \overline{0})$ – w is captured by the quantifier.

(g) $x + y$ for z in $\forall w(x + z \doteq \overline{0}) \wedge \exists y(z \doteq x)$.

$x + y$ is not free for z in $\forall w(x + z \doteq \overline{0}) \wedge \exists y(z \doteq x)$ - y is captured by the $\exists y$ quantifier.

(h) $x + y$ for z in $\forall u(u \doteq v) \rightarrow \forall z(z \doteq y)$.

$x + y$ is free for z in $\forall u(u \doteq v) \rightarrow \forall z(z \doteq y)$, and the resulting term is $\forall u(u \doteq v) \rightarrow \forall z(z \doteq y)$.

Note that van Dalen's definition of "is free for" apparently implies that $x + y$ is **not** free for z, while it clearly should be.

Section 2.4

Question 1: Let $\mathcal{N} = \langle \mathbb{N}, +, \cdot, S, 0 \rangle$ and L be a language of type $\langle -; 2, 2, 1; 1 \rangle$.

(In the following, we assume the natural, obvious interpretation. We use the language from 2.3.1.2.)

1. Give two distinct terms t in L such that $t^{\mathcal{N}} = 5$.
   $\overline{5}$ and $Plus(\overline{2}, \overline{3})$.
2. Show that for each natural number $n \in \mathbb{N}$ there is a term t such that $t^{\mathcal{N}} = n$.
   Proof by induction over $\mathbb{N}$.
   Base case: $\overline{0}^{\mathcal{N}} = 0$.
   Inductive Hypothesis: $\overline{n}^{\mathcal{N}} = n$.
   Inductive Argument: Assuming $\overline{n}^{\mathcal{N}} = n$, then let $Succ(\overline{n})^{\mathcal{N}}$ denote $n + 1 (= S(n))$.
3. Show that for each $n \in \mathbb{N}$ there are infinitely many terms t such that $t^{\mathcal{N}} = n$.
   Assume (for the purposes of contradiction) that there are only finitely many terms that denote n. Using any reasonable metric (e.g. the *rank* metric on p12, extended in the obvious way to predicate calculus), choose (one of) the largest terms in this set and call it t. Define $t' \equiv Plus(t, \overline{0})$, so $t'^{\mathcal{N}} = n$ and t' is larger than t, contradicting our maximality assumption.
   Alternative proof: interpret "infinitely many terms" as: for each natural number n there are terms $t_0, t_1, \ldots$ such that $t_i^{\mathcal{N}} = n$, for all $i \in \mathbb{N}$.
   According to (2) above, for any $n \in \mathbb{N}$ we have a term t such that $t^{\mathcal{N}} = n$. We then define t_i as $t_0 = t$ and $t_{i+1} = Plus(t_i, \overline{0})$, and prove $t_i^{\mathcal{N}} = n$ by induction on i:
   Base case: $t_0^{\mathcal{N}} = t^{\mathcal{N}} = n$
   Inductive Hypothesis: $t_i^{\mathcal{N}} = n$
   Inductive Argument: $t_{i+1}^{\mathcal{N}} = Plus(t_i, \overline{0})^{\mathcal{N}} = t_i^{\mathcal{N}} + \overline{0}^{\mathcal{N}} =$ (by Inductive Hypothesis) $= n + \overline{0}^{\mathcal{N}} = n + 0 = n$

Question 4: Which cases of **Lemma 2.4.5** remain correct if we consider formulas in general?

Note that if φ has free variable x, then $\mathcal{U} \models \varphi$ is interpreted to mean $\mathcal{U} \models \forall x.\varphi$ (p71).

This question is really asking whether it is valid to move $\forall$ around in formulas in particular ways. Indeed, many boil down to the failure of $\forall$ and $\neg$ to commute.

Lemma 2.4.5:

1. $\mathcal{U} \models \varphi \wedge \psi \Leftrightarrow \mathcal{U} \models \varphi$ and $\mathcal{U} \models \psi$
   Holds.
   To show this for arbitrary formulas φ, ψ, we need to show:
   $$\mathcal{U} \models \forall z_1, \ldots z_k.\varphi \wedge \psi \Leftrightarrow \mathcal{U} \models \forall z_1, \ldots z_k.\varphi \text{ and } \mathcal{U} \models \forall z_1, \ldots z_k.\psi$$
   where $z_1, \ldots z_k$ are all free variables of φ and ψ:
   $$\begin{aligned} & \mathcal{U} \models \forall z_1, \ldots z_k.\varphi \wedge \psi \\ \Leftrightarrow\ & [\text{by definition of } \models] \\ & [\![\forall z_1, \ldots z_k.\varphi \wedge \psi]\!]_{\mathcal{U}} = 1 \\ \Leftrightarrow\ & [\text{by definition of } [\![\ldots]\!] \text{ and substitution}] \\ & [\![(\varphi \wedge \psi)[\overline{a}_1/z_1] \ldots [\overline{a}_k/z_k]]\!]_{\mathcal{U}} = 1 \text{ for all } a_1, \ldots a_k \in |\mathcal{U}| \\ \Leftrightarrow\ & [\text{by definition of substitution}] \\ & [\![\varphi[\overline{a}_1/z_1] \ldots [\overline{a}_k/z_k] \wedge \psi[\overline{a}_1/z_1] \ldots [\overline{a}_k/z_k]]\!]_{\mathcal{U}} = 1 \text{ for all } a_1, \ldots a_k \in |\mathcal{U}| \\ \Leftrightarrow\ & [\text{by definition of } [\![\ldots]\!]] \end{aligned}$$

$$\begin{aligned}
& [\![\varphi[\overline{a}_1/z_1]\dots[\overline{a}_k/z_k]]\!]_{\mathcal{U}} = 1 \text{ and } [\![\psi[\overline{a}_1/z_1]\dots[\overline{a}_k/z_k]]\!]_{\mathcal{U}} = 1 \text{ for all } a_1,\dots a_k \in |\mathcal{U}| \\
\Leftrightarrow\ & [\text{by definition of } [\![\dots]\!]] \\
& [\![\forall z_1,\dots z_k.\varphi]\!]_{\mathcal{U}} = 1 \text{ and } [\![\forall z_1,\dots z_k.\psi]\!]_{\mathcal{U}} = 1 \\
\Leftrightarrow\ & [\text{by definition of } \models] \\
& \mathcal{U} \models \forall z_1,\dots z_k.\varphi \text{ and } \mathcal{U} \models \forall z_1,\dots z_k.\psi
\end{aligned}$$

2. $\mathcal{U} \models \varphi \vee \psi \Leftrightarrow \mathcal{U} \models \varphi$ or $\mathcal{U} \models \psi$
   Fails.
   Counterexample: Let $\varphi \equiv P(x)$ and $\psi \equiv \neg P(x)$, and $\mathcal{U}$ be a structure with two elements (say 0 and 1), where $[\![P(\overline{0})]\!]_{\mathcal{U}} = 1$ and $[\![P(\overline{1})]\!]_{\mathcal{U}} = 0$. We have $\mathcal{U} \models \forall x(P(x) \vee \neg P(x))$ but $\mathcal{U} \not\models \forall x.P(x)$ and $\mathcal{U} \not\models \forall x.\neg P(x)$.
3. $\mathcal{U} \models \neg\varphi \Leftrightarrow \mathcal{U} \not\models \varphi$
   Fails.
   Counterexample: Let $\varphi \equiv P(x)$, $[\![P(\overline{0})]\!] = 1$ and $[\![P(\overline{x})]\!] = 0$ for all other x in a structure with at least two elements. We have that $\mathcal{U} \not\models \forall x.P(x)$ and $\mathcal{U} \not\models \forall x.\neg P(x)$.
4. $\mathcal{U} \models \varphi \to \psi \Leftrightarrow (\mathcal{U} \models \varphi \Rightarrow \mathcal{U} \models \psi)$
   Fails.
   Counterexample: Let $\varphi \equiv P(x), \psi \equiv Q(x)$, $[\![P(\overline{0})]\!] = 1$ and $[\![P(\overline{x})]\!] = 0$ for all other $x \in |\mathcal{U}|$, and $[\![Q(\overline{x})]\!] = 0$ for all $x \in |\mathcal{U}|$. For the ($\Leftarrow$) case, we have that $\mathcal{U} \not\models \forall x.P(x)$, so the RHS is trivially satisfied, but $\mathcal{U} \not\models \forall x.P(x) \to Q(x)$.
5. $\mathcal{U} \models \varphi \leftrightarrow \psi \Leftrightarrow (\mathcal{U} \models \varphi \Leftrightarrow \mathcal{U} \models \psi)$
   Fails (very similar to the previous case).
6. $\mathcal{U} \models \forall x.\varphi \Leftrightarrow \mathcal{U} \models \varphi[\overline{a}/x]$, for all $a \in |\mathcal{U}|$
   Holds.
   Assume $z_1,\dots z_k$ are all free variables of $\forall x.\varphi$:

$$\begin{aligned}
& \mathcal{U} \models \forall z_1 \dots z_k.\forall x.\varphi \\
\Leftrightarrow\ & [\text{by definition of } \models] \\
& [\![\forall z_1 \dots z_k.\forall x.\varphi]\!]_{\mathcal{U}} = 1 \\
\Leftrightarrow\ & [\text{by definition of } [\![\dots]\!] \text{ and substitution}] \\
& [\![\varphi[\overline{a}_1/z_1]\dots[\overline{a}_k/z_k][\overline{a}/x]]\!]_{\mathcal{U}} = 1 \text{for all } a, a_1,\dots a_k \in |\mathcal{U}| \\
\Leftrightarrow\ & [\text{Observe that in this case the order of the substitutions does not matter}] \\
& [\![\varphi[\overline{a}/x][\overline{a}_1/z_1]\dots[\overline{a}_k/z_k]]\!]_{\mathcal{U}} = 1 \text{ for all } a, a_1,\dots a_k \in |\mathcal{U}| \\
\Leftrightarrow\ & [\text{by definition of } [\![\dots]\!]] \\
& [\![\forall z_1 \dots z_k.\varphi[\overline{a}/x]]\!]_{\mathcal{U}} = 1 \text{ for all } a \in |\mathcal{U}| \\
\Leftrightarrow\ & [\text{by definition of } \models] \\
& \mathcal{U} \models \forall z_1 \dots z_k.\varphi[\overline{a}/x] \text{ for all } a \in |\mathcal{U}|
\end{aligned}$$

7. $\mathcal{U} \models \exists x.\varphi \Leftrightarrow \mathcal{U} \models \varphi[\overline{a}/x]$, for some $a \in |\mathcal{U}|$
   Fails. Intuitively, the quantifiers (the explicit $\exists$ and implicit $\forall$) are swapped.
   Counterexample: Let $\varphi \equiv P(x,y)$. Take a two-element structure $\mathcal{U}$ with the following interpretation:

$$\begin{aligned}
[\![P(\overline{0},\overline{0})]\!] &= 0 \\
[\![P(\overline{0},\overline{1})]\!] &= 1 \\
[\![P(\overline{1},\overline{0})]\!] &= 1 \\
[\![P(\overline{1},\overline{1})]\!] &= 0
\end{aligned}$$

   We have that $\mathcal{U} \models \forall y \exists x.P(x,y)$, but not that $\mathcal{U} \models \forall y.P(\overline{a},y)$ for some $a \in |\mathcal{U}|$.

Question 5: For sentences σ we have $\mathcal{U} \models \sigma$ or $\mathcal{U} \models \neg\sigma$. Show that this does not hold for φ with $FV(\varphi) \neq \emptyset$. Show that not even for sentences does "$\models \sigma$ or $\models \neg\sigma$" hold.

First part: Let $\varphi \equiv \neg P(x)$, and take a structure $\mathcal{U}$ with two elements, $[\![P(\overline{0})]\!] = 0$ and $[\![P(\overline{1})]\!] = 1$. Then $\mathcal{U} \not\models \forall x.\neg P(x)$ and $\mathcal{U} \not\models \forall x.\neg\neg P(x)$.

Second part: As we are quantifying over all suitable structures, all we need to do is find a σ where σ fails in one structure, and $\neg\sigma$ fails in another.

Consider $\sigma \equiv \forall x.P(x)$. First structure: two elements, $[\![P(x)]\!] = 0$ for both values of x. Second structure: two elements, $[\![P(x)]\!] = 1$ for both elements.

Question 7: Show that $\mathcal{U} \models \varphi \Rightarrow \mathcal{U} \models \psi$ for all $\mathcal{U}$ implies $\models \varphi \Rightarrow \models \psi$, but not vice versa.

($\Rightarrow$) Assume that for all $\mathcal{U}$, $\mathcal{U} \models \varphi \Rightarrow \mathcal{U} \models \psi$ (call this assumption (1)). We now need to show that $\models \varphi \Rightarrow \models \psi$.

Assume $\models \varphi$. By definition this means that for all $\mathcal{A}$, we know $\mathcal{A} \models \varphi$. By (1), it follows that $\mathcal{A} \models \psi$ for all $\mathcal{A}$. By definition, again, we then have that $\models \psi$. QED.

($\not\Leftarrow$) To show that the implication from right to left does not hold, we need to find φ and ψ such that the RHS holds, and the LHS does not. So, take φ to be the atom P, and ψ to be $\neg P$ and show that then the RHS holds (1), but the LHS does not (2).

1. Since there are structures where P is false, we know that $\not\models P$. Therefore, the implication $\models P \Rightarrow \models \neg P$ holds.
2. We need to find a structure $\mathcal{U}$ such that $\mathcal{U} \models P \not\Rightarrow \mathcal{U} \models \neg P$. Take $\mathcal{U}$ to be any structure such that $[\![P]\!]_{\mathcal{U}} = 1$. Then $\mathcal{U} \models P$, but $\mathcal{U} \not\models \neg P$, hence $\mathcal{U} \models P \not\Rightarrow \mathcal{U} \models \neg P$.