Logic in Computer Science

Another presentation of natural deduction

We use the letters $\Gamma, \Delta, \ldots$ for *sequences* of formulae of the form $\phi_1, \ldots, \phi_n$ (n may be 0 in which case the sequence is empty). If Γ is $\phi_1, \ldots, \phi_n$ we write Γ, ϕ for $\phi_1, \ldots, \phi_n, \phi$.

We give another definition of $\Gamma \vdash \phi$, by inference rules. The axioms are

$$\frac{}{\Gamma \vdash \phi}$$

whenever ϕ is one of the formula $\phi_1, \ldots, \phi_n$ (notice that it may appear several times). For instance

$$p, q \vdash p \qquad p, q, p \vdash q \qquad p, q, p \vdash p$$

We have then the following rules

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}\ (\to i) \qquad \frac{\Gamma \vdash \phi \to \psi \quad \Gamma \vdash \phi}{\Gamma \vdash \psi}\ (\to e)$$

$$\frac{\Gamma, \phi \vdash \bot}{\Gamma \vdash \neg\phi}\ (\neg i) \qquad \frac{\Gamma \vdash \neg\phi \quad \Gamma \vdash \phi}{\Gamma \vdash \bot}\ (\neg e)$$

$$\frac{\Gamma \vdash \phi_1}{\Gamma \vdash \phi_1 \vee \phi_2}\ (\vee i) \qquad \frac{\Gamma \vdash \phi_2}{\Gamma \vdash \phi_1 \vee \phi_2}\ (\vee i) \qquad \frac{\Gamma \vdash \phi_1 \vee \phi_2 \quad \Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma \vdash \psi}\ (\vee e)$$

$$\frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_1}\ (\wedge e) \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_2}\ (\wedge e) \qquad \frac{\Gamma \vdash \phi_1 \quad \Gamma \vdash \phi_2}{\Gamma \vdash \phi_1 \wedge \phi_2}\ (\wedge i)$$

This defines intuitionistic logic. In order to get classical logic, we have to add the law of double negation elimination

$$\frac{\Gamma \vdash \neg\neg\phi}{\Gamma \vdash \phi}$$

Here is for instance a derivation of $\vdash p \to (q \to p)$:

1. $p, q \vdash p$ axiom
2. $p \vdash q \to p$ by $\to i$
3. $\vdash p \to (q \to p)$ by $\to i$

Formally a derivation is a sequence of sequents (!) $s_1, \ldots, s_n$ such that any s_k is either an axiom or can be derived using one the rule above from some $s_i,\ i < k$. Let us give another example:

1. $p \wedge q \vdash p \wedge q$ axiom
2. $p \wedge q \vdash p$ by $\wedge e$:1
3. $p \wedge q \vdash q$ by $\wedge e$:1
4. $p \wedge q \vdash q \wedge p$ by $\wedge i$:2,3

Here is an instance of a *derived* (or *admissible*) rule:

$$\frac{\Gamma, \neg\phi \vdash \bot}{\Gamma \vdash \phi}$$

and here is the derivation

1. $\Gamma, \neg\phi \vdash \bot$ assumption
2. $\Gamma \vdash \neg\neg\phi$ $\neg i$
3. $\Gamma \vdash \phi$ $\neg\neg e$

The advantage of this presentation is that we can give a nicer proof of the soundness Theorem.

Theorem: *If* $\Gamma \vdash \phi$ *then* $\Gamma \models \phi$

We prove this *by course of value induction.* If we have a derivation $\Gamma_1 \vdash \phi_1, \dots, \Gamma_n \vdash \phi_n$ then we have also $\Gamma_1 \models \phi_1, \dots, \Gamma_n \models \phi_n$. This is direct if $\Gamma_k \models \phi_k$ is an axiom, because then ϕ_k appears in the sequence Γ_k. If we derive $\Gamma_k \models \phi_k$ from previous sequents, the Theorem holds by induction. We have to look at all possible rules. I give only two examples:

If we derive $\Gamma_k \vdash \phi_k$ by $\rightarrow e$ then we have $i, j < k$ with $\phi_j = \phi_i \rightarrow \phi_k$ and $\Gamma_k = \Gamma_i = \Gamma_j$. By induction hypothesis we have $\Gamma_i \models \phi_i$ and $\Gamma_j \models \phi_j$. So $\Gamma_k \models \phi_i$ and $\Gamma_k \models \phi_i \rightarrow \phi_k$. If we have a valution ρ that makes T all formulae in Γ_k then ϕ_i and $\phi_i \rightarrow \phi_k$ get the value T. So ϕ_k gets the value T. We have shown $\Gamma_k \models \phi_k$ as required.

If we derive $\Gamma_k \vdash \phi_k$ by $\wedge i$ then we have $i, j < k$ with $\phi_k = \phi_i \wedge \phi_j$ and $\Gamma_k = \Gamma_i = \Gamma_j$. By induction hypothesis we have $\Gamma_i \models \phi_i$ and $\Gamma_j \models \phi_j$. So $\Gamma_k \models \phi_i$ and $\Gamma_k \models \phi_j$. If we have a valution ρ that makes T all formulae in Γ_k then ϕ_i and ϕ_j get the value T. So $\phi_k = \phi_i \wedge \phi_j$ gets the value T. We have shown $\Gamma_k \models \phi_k$ as required.

Application of the soundness Theorem

Theorem: *Propositional calculus is* consistent*; we cannot have both* $\vdash \phi$ and $\vdash \neg\phi$

Indeed it is clear that we cannot have both $\models \phi$ and $\models \neg\phi$

The soundness Theorem is also useful to show that a formula *cannot be* proved. For instance, we don't have

$$D \rightarrow \neg G,\ W \rightarrow D \vdash \neg W \rightarrow G \quad (*)$$

because the assignement $W = \mathit{True}$, $G = \mathit{False}$ makes the premisses True and the conclusion False (independently of the assignement to the formula D).

Here is an example of a reformulation of (*), which may show that it is not always so easy to guess if an argument is correct or not: "it is not good if I am depressed, and if I watch the news I am depressed; hence it is good that I don't watch the news".

Natural deduction for first-order logic

Here are the rules for universal quantification

$$\frac{\Gamma \vdash \forall x.\phi}{\Gamma \vdash \phi[x/t]}\ (\forall e)$$

provided t is free for x in ϕ and

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \forall x.\phi}\ (\forall i)$$

provided x is not free in *any* formula of Γ.

The rules for existential quantification are

$$\frac{\Gamma \vdash \phi[x/t]}{\Gamma \vdash \exists x.\phi}\ (\exists i)$$

provided t is free for x in ϕ and

$$\frac{\Gamma \vdash \exists x.\phi \qquad \Gamma, \phi \vdash \psi}{\Gamma \vdash \psi}\ (\exists e)$$

provided x is not free in ψ and not free in any formula of Γ.