Krull Dimension

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Zariski spectrum

Any element of the Zariski lattice is of the form $D(a_1, \ldots, a_n) = D(a_1) \vee \cdots \vee D(a_n)$. We have seen that $D(a, b) = D(a + b)$ if $D(ab) = 0$

In general we cannot write $D(a_1, \ldots, a_n)$ as $D(a)$ for *one* element a

We can ask: what is the least number m such that *any* element of Zar(R) can be written on the form $D(a_1, \ldots, a_m)$. An answer is given by the following version of *Kronecker's Theorem*: this holds if Kdim $R < m$

Krull dimension of a ring

The *Krull dimension* of a ring is defined to be the maximal length of proper chain of prime ideals.

In fact, one can give a purely algebraic definition of the Krull dimension of a ring

Inductive definition of dimension of spectral spaces/distributive lattice: Kdim $X \leqslant n$ iff for any compact open U we have Kdim $Bd(U) < n$ (cf. Menger-Urysohn definition of dimension)

To be zero-dimensional is to be a Boolean lattice

Krull dimension of a lattice

If L is a lattice, we say that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are *(n-)complementary* iff

$$u_1 \vee v_1 = 1,\ u_1 \wedge v_1 \leqslant u_2 \vee v_2, \ldots, u_{n-1} \wedge v_{n-1} \leqslant u_n \vee v_n,\ u_n \wedge v_n = 0$$

For $n = 1$: we get that u_1 and v_1 are complement

Proposition: Kdim $L < n$ *iff any n-sequence of elements has a complementary sequence*

Krull dimension of a lattice

What is important here is the logical complexity

Distributive lattice: equational theory

The notion of complementary sequence is a (first-order) coherent notion

Complementary sequence

If a_1, a_2 and b_1, a_2 have a complementary sequence then so has $a_1 \vee b_1, a_2$ and $a_1 \wedge b_1, a_2$

If a_1, a_2 and a_1, b_2 have a complementary sequence then so has $a_1, a_2 \vee b_2$ and $a_1, a_2 \wedge b_2$

In this way to ensure the existence of complementary sequence it is enough to look only at elements in a generating subset of the lattice

Krull dimension of a ring

Kdim $R < n$ is defined as Kdim $(\mathsf{Zar}(R)) < n$

Proposition: Kdim $R < n$ *iff for any sequence* $a_1, \ldots, a_n$ in R *there exists a sequence* $b_1, \ldots, b_n$ in R *such that, in* Zar(R), *we have*

$$D(a_1, b_1) = 1,\ D(a_1 b_1) \leqslant D(a_2, b_2), \ldots, D(a_{n-1} b_{n-1}) \leqslant D(a_n, b_n),\ D(a_n b_n) = 0$$

This is a *first-order* condition in the multi-sorted language of rings and lattices

Example: Kronecker's theorem

Kronecker in section 10 of

Grundzüge einer arithmetischen Theorie der algebraischen Grössen. *J. reine angew. Math.* 92, 1-123 (1882)

proves a theorem which is now stated in the following way

An algebraic variety in $\mathbb{C}^n$ *is the intersection of* $n+1$ *hypersurfaces*

Kronecker's Theorem

Theorem: *If* Kdim $R < n$ *then for any* $b_0, b_1, \ldots, b_n$ *there exist* $a_1, \ldots, a_n$ *such that* $D(b_0, \ldots, b_n) = D(a_1, \ldots, a_n)$

This is a (non Noetherian) generalisation of Kronecker's Theorem

For each fixed n this is a first-order tautology. So, by the completeness Theorem for first-order logic, it has a first-order proof

It says that if Kdim $R < n$ then we can write any elements of the Zariski lattice on the form $D(a_1, \ldots, a_n)$

Kronecker's Theorem

In particular if R is a polynomial ring $k[X_1, \ldots, X_m]$ with $m < n$ then this says that given $n + 1$ polynomials we can find n polynomials that have the *same* set of zeros in an arbitrary algebraic closure of k

Kronecker's Theorem

This concrete proof/algorithm, is *extracted* from R. Heitmann "*Generating non-Noetherian modules efficiently*" Michigan Math. J. 31 (1984), 167-180

Though seeemingly unfeasible (use of prime ideals, topological arguments on the Zariski spectrum) this paper contains implicitely a clever and simple algorithm which can be instantiated for polynomial rings

Kronecker's Theorem

Kronecker's Theorem is direct from the existence of complementary sequence

Lemma: *If X, Y are complementary sequence then for any element a we have* $D(a, X) = D(X - aY)$

Since we have $D(a, X - aY) = D(a, X)$ it is enough to show $D(a) \leqslant D(X - aY)$

$D(x_1 - ay_1, x_2 - ay_2) = D(x_1 - ay_1, x_2, ay_2)$ since $D(x_2 y_2) = 0$

$D(x_1 - ay_1, x_2, y_2) = D(x_1, ay_1, x_2, y_2) = D(a)$ since $D(x_1 y_1) \leqslant D(x_2, y_2)$

Forster's Theorem

We say that a sequence $s_1, \ldots, s_l$ of elements of a commutative ring R is *unimodular* iff $D(s_1, \ldots, s_l) = 1$ iff $R = < s_1, \ldots, s_l >$

If M is a matrix over R we let $\Delta_n(M)$ be the ideal generated by all the $n \times n$ minors of M

Theorem: *Let M be a matrix over a commutative ring R. If $\Delta_n(M) = 1$ and* Kdim $R < n$ *then there exists an unimodular combination of the column vectors of M*

This is a non Noetherian version of Forster's 1964 Theorem

Forster's Theorem

We get a first-order (constructive) proof.

It can be interpreted as an algorithm which produces the unimodular combination.

The motivation for this Theorem comes from differential geometry

If we have a vector bundle over a space of dimension d and all the fibers are of dimension r then we can find $d+r$ generators for the module of global sections

Forster's Theorem

The proof relies on the following consequence of Cramer formulae

Proposition: *If P is a $n \times n$ matrix of determinant δ and of adjoint matrix $\tilde{P}$ then we have $D(\delta X - \tilde{P}Y) \leqslant D(PX - Y)$ for arbitrary column vectors X, Y in $R^{n \times 1}$*

Corollary: *If P P is a $n \times n$ matrix of determinant δ and $X, \tilde{P}Y$ are complementary then $D(\delta) \leqslant D(P(\delta X) - Y)$*

Serre's Spliting-Off Theorem

This is the special case where the matrix is idempotent

The existence of a unimodular combination of the column in this case has the following geometrical intuition.

We have countinuous family of vector spaces over a base space. If the dimension of each fibers of a fibre bundle is $>$ the dimension of the base space, one can find a non vanishing section

This is not the case in general: Moebius strip, tangent bundle of S^2

Vector bundles are represented as finitely generated projective modules

Elimination of noetherian hypotheses

Kronecker's Theorem, Forster's Theorem were first proved with the hypothesis that the ring R is noetherian

The fact that we can eliminate this hypothesis is remarkable

An example of a first-order statement for which we *cannot* eliminate this hypothesis is the Regular Element Theorem which says that if $I = <a_1, \ldots, a_n>$ is regular (that is $uI = 0$ implies $u = 0$) then we can find a regular element in I.