Universe of Bishop sets

Introduction

The goal of this note is to present a self-contained presentation of the cubical set model and a natural sub-presheaf of the universe which corresponds to the notion of Bishop sets[1], where two paths having the same endpoints are necessarily equal for *judgemental* equality. We show that this sub-presheaf is fibrant and it is actually closed by the Kan structure of the universe.

All we present can be formalized in Aczel's system CZFu (constructive version of ZFC with one Grothendieck universe), and what we present of the cubical set model has been formalized by Mark Bickford in NuPrl.

1 Base category, fibrations and cofibrations

We write $I, J, K, \ldots$ the objects of a given small category $\mathcal{C}$.

We write $A, B, \ldots, \Gamma, \Delta, \ldots$ for presheaves over $\mathcal{C}$. (A presheaf A is given by a collection of sets $A(I)$ with restriction maps $A(I) \to A(J)$ sending u to uf for $f : J \to I$.) We use the same notation for an object I and the presheaf it represents.

We assume given a special presheaf $\mathbb{I}$ which has a structure of distributive lattice with an involution (a.k.a. de Morgan algebra). We also assume that $J \times \mathbb{I}$ is always representable and we have a functor $J \longmapsto J^+$ on the base category so that J^+ represents $J \times \mathbb{I}$. We have two maps $e_0, e_1 : J \to J^+$ that are sections of the projection $p : J^+ \to J$.

We also assume given a subobject $\mathbb{F}$ of the suboject classifier which is a sub-lattice. Any map $\psi : A \to \mathbb{F}$ defines a subpresheaf $A|\psi \subseteq A$ where $(A|\psi)(I)$ is the subset of element ρ in $\Gamma(I)$ such that $\psi\rho = 1$ in $\mathbb{F}(I)$. We assume maps $\delta_0, \delta_1 : A \times \mathbb{I} \to \mathbb{F}$ which classify respectively the two face maps $e_0, e_1 : A \to A \times \mathbb{I}$. For proving fibrancy of the universe (and that fact that it is univalent) we also assume that the constant map $\mathbb{F} \to \mathbb{F}^{\mathbb{I}}$ has a right adjoint $\forall : \mathbb{F}^{\mathbb{I}} \to \mathbb{F}$.

If we have $\sigma : A \to B$ and $\psi : B \to \mathbb{F}$ then σ induces a map $A|\psi\sigma \to B|\psi$, that sends u in $(A|\psi)(I)$ to $\sigma_I u$. We may write simply $\sigma : A|\psi\sigma \to B|\psi$ for this induced map.

We say that a map is a *cofibration* if, and only if, it is classified by $\mathbb{F}$.

A (generalised) *open box* $\mathsf{b}(A, \psi) \subseteq A \times \mathbb{I}$ is the subpresheaf determined by $\delta_0 \vee \psi p : A \times \mathbb{I} \to \mathbb{F}$ for some $\psi : A \to \mathbb{F}$.

A *fibration* is a map that has the right lifting property w.r.t. any open box[2]. A *trivial fibration* is a map which has the right lifting property w.r.t. any cofibration.

2 Dependent types

If Γ is a cubical set, we can consider its *category of elements* $\int \Gamma$: an object is of the form (I, ρ) with ρ in $\Gamma(I)$ and a map $f : (J, \nu) \to (I, \rho)$ is a map $f : J \to I$ such that $\nu = \rho f$ in $\Gamma(J)$.

A *dependent type* on Γ, notation $\Gamma \vdash A$, is a presheaf on the category of elements of Γ.

If $\sigma : \Delta \to \Gamma$ then σ determines a functor $\int \Delta \to \int \Gamma$ sending (I, ν) to $(I, \sigma\nu)$ and $\Gamma \vdash A$ determines by composition a dependent type $\Delta \vdash A\sigma$.

[1]Intuitively, a Bishop set can be seen as a (directed) graph of an equivalence relation, where there is at most one edge between two nodes.

[2]This corresponds to what Cisinski calls 《naive》 fibrations, but in our case they coincide with fibrations.

If we have $\Gamma \vdash A$ we can define a new cubical set $\Gamma.A$ by taking $(\Gamma.A)(I)$ to be the set of elements ρ, u with ρ in $\Gamma(I)$ and u in $A(I,\rho)$ and $(\rho,u)f = (\rho f, uf)$. We have a canonical map $p_A : \Gamma.A \to \Gamma$ defined by $p_A(\rho,u) = \rho$.

We define $\Gamma \vdash u : A$ to mean that u is a family of elements $u(I,\rho)$ in $A(I,\rho)$ such that $u(I,\rho)f = u(J,\rho f)$ if $f : J \to I$. If $\sigma : \Delta \to \Gamma$ we can define $\Delta \vdash u\sigma : A\sigma$ by $u\sigma(I,\rho) = u(I,\sigma\rho)$.

It is convenient to identify the set of maps $I \to \Gamma$ with the set $\Gamma(I)$. If $\rho : I \to \Gamma$ and $\Gamma \vdash A$ we can consider the presheaf $I \vdash A\rho$ and the set $A(I,\rho)$ and we have $A\rho(I,1) = A(I,\rho)$. The set of sections $I \vdash u : A\rho$ can be identified with the set $A(I,\rho)$.

In the case $\Delta \subseteq \Gamma$ and $\Gamma \vdash A$ we may write $\Delta \vdash A$ omitting the canonical inclusion map. Similarly if $\Gamma \vdash u : A$ then we still write $\Delta \vdash u : A$ if $\Delta \subseteq \Gamma$ omitting the canonical inclusion map.

Though the presheaf category on $\int \Gamma$ and the slice category over Γ are equivalent, it is important to distinguish them to be able to state results with strict equality (which is crucial to get a model of type theory in a simple way without coherence issues).

If $\Gamma \vdash A$ and $\Gamma \vdash a_0 : A$ and $\Gamma \vdash a_1 : A$ we define $\Gamma \vdash \mathsf{Path}\ A\ a_0\ a_1$ by

$$(\mathsf{Path}\ A\ a_0\ a_1)(I,\rho) = \{\omega \in A(I^+, \rho p) \mid \omega e_0 = a_0(I,\rho) \wedge \omega e_1 = a_1(I,\rho)\}$$

and the restriction map $(\mathsf{Path}\ A\ a_0\ a_1)(I,\rho) \to (\mathsf{Path}\ A\ a_0\ a_1)(J,\rho f)$ for $f : J \to I$, sends ω to ωf^+.

3 Trivial fibrations and filling structures

Definition 3.1 A filling structure for $\Gamma \vdash A$ is an operation $I^+ \vdash C_A(\rho,\psi,u) : A\rho$ given $\rho : I^+ \to \Gamma$ and $\psi : I \to \mathbb{F}$ and $\mathsf{b}(I,\psi) \vdash u : A$ satisfying $\mathsf{b}(I,\psi) \vdash C_A(\rho,\psi,u) = u : A$ *together with* the uniformity condition that if $f : J \to I$ then

$$J^+ \vdash C_A(\rho,\psi,u)f^+ = C_A(\rho f^+, \psi f, uf^+) : A\rho f^+$$

Notice that $f : J \to I$ induces a map $f^+ : \mathsf{b}(J,\psi f) \to \mathsf{b}(I,\psi)$.

Similarly a trivial fibration structure is giving by explicit operations $I \vdash e_A(\rho,\psi,u) : A\rho$ with $\psi : I \to \mathbb{F}$ and $\rho : I \to \Gamma$ and $I,\psi \vdash u : A$ satisfying $I,\psi \vdash e_A(\rho,\psi,u) = u : A$ together with the uniformity condition that if $f : J \to I$ then

$$J \vdash e_A(\rho,\psi,u)f = e_A(\rho f, \psi f, uf) : A\rho f$$

Theorem 3.2 *A dependent type $\Gamma \vdash A$ has a filling structure (resp. trivial fibration structure) if, and only is, p_A is a fibration (resp. a trivial fibration).*

Definition 3.3 A composition structure for $\Gamma \vdash A$ is an operation $I \vdash c_A(\rho,\psi,u) : A\rho e_1$ given $\rho : I^+ \to \Gamma$ and $\psi : I \to \mathbb{F}$ and $\mathsf{b}(I,\psi) \vdash u : A\rho$ satisfying $I,\psi \vdash c_A(\rho,\psi,u) = ue_1 : A\rho e_1$ *together with* the uniformity condition that if $f : J \to I$ then

$$J^+ \vdash c_A(\rho,\psi,u)f = c_A(\rho f^+, \psi f, uf^+) : A\rho e_1 f$$

If c_A is a composition structure on $\Gamma \vdash A$ and $\sigma : \Delta \to \Gamma$ then we define a composition structure $c_A\sigma$ on $\Delta \vdash A\sigma$ by $I^+ \vdash c_A\sigma(\nu,\psi,u) = c_A(\sigma\nu,\psi,u) : A\sigma\nu$.

Theorem 3.4 *The set of composition structures on $\Gamma \vdash A$ is a* retract *of the set of filling structures on $\Gamma \vdash A$.*

4 Universe

We consider one (constructive) Grothendieck universe $\mathcal{U}$ and define a corresponding cubical set U by taking $U(I)$ to be the set of all pairs (A, c_A) where A is a $\mathcal{U}$-dependent type $I \vdash A$ and c_A is a composition structure on $I \vdash A$. If $f : J \to I$ we define $(A, c_A)f$ to be Af, c_Af.

We define $U \vdash El$ by taking $El(I, A, c_A)$ to be the set of all sections $I \vdash u : A$. The element (A, c_A) can be seen as a map $I \to U$ and we have $I \vdash El(A, c_A) = A$.

Theorem 4.1 *$U \vdash El$ has a canonical composition structure c_E such that if $\Gamma \vdash A$ is a $\mathcal{U}$-dependent type with a composition structure c_A, there exists* a unique *map $|A| : \Gamma \to U$ such that $El|A| = A$ and $c_E|A| = c_A$ (*strict *equalities).*

Proof. Any map $I^+ \to U$ is of the form (A, c_A) with $I^+ \vdash A$ and c_A a corresponding composition structure. We define then

$$I^+ \vdash c_E((A, c_A), \psi, u) = c_A(1, \psi, u) : El(A, c_A)$$

given ψ in $\mathbb{F}(I)$ and $\mathsf{b}(I, \psi) \vdash u : El(A, c_A) = A$. We can then check, if $f : J \to I$

$$c_E((A, c_A), \psi, u)f^+ = c_A(1, \psi, u)f^+ = c_A(f^+, \psi f, uf^+) = c_A f^+(1, \psi f, uf^+) = c_E((A, c_A)f^+, \psi f, uf^+)$$

If $\Gamma \vdash A$ is a $\mathcal{U}$-dependent type with a composition structure c_A, we define $|A| : \Gamma \to U$ by taking $|A|\rho = A\rho, c_A\rho$. We then have $El|A|\rho = El(A\rho, c_A\rho) = A\rho$ and

$$c_E|A|(\rho, \psi, u) = c_E(|A|\rho, \psi, u) = c_E((A\rho, c_A\rho), \psi, u) = c_A\rho(1, \psi, u) = c_A(\rho, \psi, u)$$

hence $El|A| = A$ and $c_E|A| = c_A$. □

5 Strict propositions and Bishop sets

We say that $\Gamma \vdash A$ is a *strict proposition*, notation

$$\Gamma \vdash A \ \mathsf{sprop}$$

if, and only if, for any ρ in $\Gamma(I)$ the set $A(I, \rho)$ is a sub-singleton.

We say that $\Gamma \vdash A$ is a *Bishop set*, notation

$$\Gamma \vdash A \ \mathsf{bset}$$

if, and only if, for any $\rho : I^+ \to \Gamma$ and any two elements u, u' in $A(I^+, \rho)$ we have $u = u'$ as soon as $ue_0 = u'e_0$ and $ue_1 = u'e_1$.

We have that $\Gamma \vdash A$ is a Bishop set if it is a strict proposition.

The following rules are valid

$$\frac{\Gamma \vdash A \ \mathsf{sprop}}{\Gamma \vdash A \ \mathsf{bset}} \qquad \frac{\Gamma \vdash A \ \mathsf{bset} \qquad \Gamma \vdash a_0 \ a_1 : A}{\Gamma \vdash \mathsf{Path} \ A \ a_0 \ a_1 \ \mathsf{sprop}} \qquad \frac{\Gamma \vdash A \ \mathsf{sprop} \qquad \Gamma \vdash a_0 \ a_1 : A}{\Gamma \vdash a_0 = a_1 : A}$$

We also have that these notions are closed by substitution: if $\sigma : \Delta \to \Gamma$ then $\Gamma \vdash A$ sprop (resp. $\Gamma \vdash A$ bset) implies $\Delta \vdash A\sigma$ sprop (resp. $\Delta \vdash A\sigma$ bset).

If $\Gamma \times \mathbb{I} \vdash A$ bset and $\Gamma \times \mathbb{I} \vdash u \ u' : A$ and $\Gamma \vdash ue_0 = u'e_0 : Ae_0$ and $\Gamma \vdash ue_1 = u'e_1 : Ae_1$ then we have $\Gamma \times \mathbb{I} \vdash u = u' : A$.

We define then two cubical sets $\mathsf{sProp} \subseteq \mathsf{bSet} \subseteq U$ by taking $\mathsf{sProp}(I)$ to be the set of all pairs A, c_A in $U(I)$ such that $I \vdash A$ is a strict proposition, and $\mathsf{bSet}(I)$ to be the set of all pairs A, c_A in $U(I)$ such that $I \vdash A$ is a Bishop set. Since $U \vdash El$ we also have by restriction $\mathsf{sProp} \vdash El$ and $\mathsf{bSet} \vdash El$.

Theorem 5.1 *If $\Gamma \vdash A$ is a $\mathcal{U}$-dependent type with a composition structure c_A, and is a strict proposition (resp. Bishop set) there exists* a unique *map $|A| : \Gamma \to \mathsf{sProp}$ (resp. $|A| : \Gamma \to \mathsf{bSet}$) such that $El|A| = A$ and $c_E|A| = c_A$ (*strict *equalities).*

We are going next to see that all these 3 universes are fibrant and the inclusion maps commute (strictly) with composition.

6 Fibrancy of the universes

The fact that the universe U is fibrant relies on the following «glueing» operation, which can be defined at the level of presheaves.

Theorem 6.1 *Given $\Gamma \vdash A$ and $\psi : \Gamma \to \mathbb{F}$ and $\Gamma|\psi \vdash T$ and*[3] *$\Gamma|\psi \vdash w : T \to A$, we can define $\Gamma \vdash \mathsf{Glue}(A, \psi, T, w)$ and $\Gamma \vdash e(A, \psi, T, w) : \mathsf{Glue}(A, \psi, T, w) \to A$ such that $\Gamma|\psi \vdash T = \mathsf{Glue}(A, \psi, T, w)$ and $\Gamma|\psi \vdash w = e(A, \psi, T, w) : T \to A$ and if $\sigma : \Delta \to \Gamma$ then*

$$\mathsf{Glue}(A, \psi, T, w)\sigma = \mathsf{Glue}(A\sigma, \psi\sigma, T\sigma, w\sigma) \qquad e(A, \psi, T, w)\sigma = e(A\sigma, \psi\sigma, T\sigma, w\sigma)$$

Proof. we define $G = \mathsf{Glue}(A, \psi, T, w)$ by defining a family of sets $G(I, \rho)$ given $\rho : I \to \Gamma$ by case on $\psi\rho$

1. if $\psi\rho = 1$ then $G(I, \rho)$ is the set $T(I, \rho)$ which is well defined since ρ is in $(\Gamma|\psi)(I)$ in this case
2. if $\psi\rho \neq 1$ then $G(I, \rho)$ is the set of pairs a, t where a is in $A(I, \rho)$ and $I|\psi\rho \vdash t : T\rho$ such that $w_{(J,\rho f)}\ t(J, f) = af$ in $A(J, \rho f)$ whenever $f : J \to I$ and $\psi\rho f = 1$.

We define then the restriction map $G(I, \rho) \to G(J, \rho f)$ for $f : J \to I$ by corresponding cases. If $\psi\rho = 1$ it is the restriction map of T. If $\psi\rho \neq 1$ and $\psi\rho f = 1$ then $(a, t)f = tf$ and finally if $\psi\rho f \neq 1$ then $(a, t)f = af, tf$. □

Theorem 6.2 *If $\Gamma \vdash A$ is a Bishop set (resp. a strict proposition) and $\psi : \Gamma \to \mathbb{F}$ and $\Gamma|\psi \vdash T$ is a Bishop set (resp. a strict proposition) and $\Gamma|\psi \vdash w : T \to A$ then $\Gamma \vdash \mathsf{Glue}(A, \psi, T, w)$ is a Bishop set (resp. a strict proposition).*

If $\Delta \vdash T$ and $\Delta \vdash A$ we write $\Delta \vdash w : T \to A$ to mean that w is a natural transformation between the two presheafs T and A on $\int \Delta$. We define the *homotopy fiber* $\Delta.A \vdash F_w$ by taking $F_w(I, \rho, u)$ for ρ in $\Delta(I)$ and u in $A(I, \rho)$ to be the set of elements (t, ω) where t is in $T(I, \rho)$ and ω an element of $A(I^+, \rho p)$ such that $\omega e_0 = w\ t$ and $\omega e_1 = u$. If $f : J \to I$ we define $(t, \omega)f = (tf, \omega f^+)$.

An *equivalence structure* c_w for the map w is then a trivial fibration structure for $\Delta.A \vdash F_w$ (this expresses that each fiber of w is contractible).

Theorem 6.3 *Given $\Gamma \vdash A$ with a composition structure c_A and $\psi : \Gamma \to \mathbb{F}$ and $\Gamma|\psi \vdash T$ with a composition structure c_T and $\Gamma|\psi \vdash w : T \to A$ with an equivalence structure c_w we can find a composition structure $\mathsf{Glue}(c_A, \psi, c_T, c_w)$ on $\Gamma \vdash \mathsf{Glue}(A, \psi, T, w)$ such that $\mathsf{Glue}(c_A, \psi, c_T, c_w)\sigma = \mathsf{Glue}(c_A\sigma, \psi\sigma, c_T, c_w\sigma)$ if $\sigma : \Delta \to \Gamma$ and $\mathsf{Glue}(c_A, \psi, c_T, c_w) = c_T$ if $\psi = 1$.*

Corollary 6.4 *The universe U has a composition structure c_U such that* bSet *and* sProp *are closed by c_U.*

Proof. Given I and $\psi : I \to \mathbb{F}$ and $\mathsf{b}(I, \psi) \vdash X : U$ we have to define $I \vdash c_U(\psi, X) : U$ such that $I, \psi \vdash c_U(\psi, X) = Xe_1 : U$. We define $I \vdash A = Xe_0$ and $I, \psi \vdash T = Xe_1$ and X defines a transport map $I, \psi \vdash w : T \to A$ which has an equivalence structure. It follows then that we get a composition structure c_G on $I \vdash \mathsf{Glue}(A, \psi, T, w)$ and we define $c_U(\psi, X) = \mathsf{Glue}(A, \psi, T, w), c_G$. Theorem 6.2 shows then that bSet and sProp are closed by c_U. □

In particular, sProp and bSet are fibrant.

It also follows from Theorems 6.3 and 6.2 that both universes sProp and bSet are univalent.

[3]This means that w is a natural transformation between the presheaves T and A on the category of elements of $\Gamma|\psi$.

7 Strict sets

We could define a notion of *strict sets*, and a corresponding notion of universe, by requiring of $\Gamma \vdash A$ that, if $\rho : I^+ \to \Gamma$ and u_0 in $A(I, \rho e_0)$, we have *at most one* ω in $A(I^+, \rho)$ such that $u_0 = \omega e_0$. A fibration $\Gamma \vdash A$ which is a strict set would correspond to the notion of *covering space* and syntactically to the *equality reflection rule*. (By contrast, we expect our notion of Bishop sets to correspond syntactically to a type system with decidable type-checking.)

Notice that, if $\Gamma \vdash A$ is a strict set, it has at most one composition structure. So in this case, to have a composition is a property and not only a structure.

We can then define a presheaf sSet with $\mathsf{sSet}(I)$ is the set of elements A, c_A in $U(I)$ such that $I \vdash A$ is a strict set. We have $\mathsf{sProp} \subseteq \mathsf{sSet} \subseteq \mathsf{bSet}$.

Let N_2 be the set $\{0, 1\}$. This defines a strict set (constant presheaf). We define a non trivial strict set $\mathbb{I} \vdash E$ corresponding to the bijection swapping 0 and 1, by taking $E(J, r)$ for r in $\mathbb{I}(J)$ to also be constantly N_2 but with a non trivial restriction map $E(J, r) \to E(K, rg)$ for $g : K \to J$ which is defined by cases: if $r = 1$ or $rg \neq 1$ it is constant, and it is the swapping map if $r \neq 1$ and $rg = 1$. It can be shown that $\mathbb{I} \vdash E$ has a composition structure.

Lemma 7.1 *If $\Gamma \vdash A$ then it is a strict set as soon as if ω, ω' are in $A(I^+, \rho)$ then $\omega e_0 = \omega' e_0$ implies $\omega e_1 = \omega' e_1$.*

Proof. Let us assume that this condition holds, and that we have $\omega e_0 = \omega' e_0$ and we prove $\omega = \omega'$. Since $\mathbb{I}$ has a lattice structure, we have $m : I^{++} \to I^+$ such that $me_1 = 1$ and $me_0 = e_0 p$. We have $\omega e_0 p = \omega' e_0 p$ and hence $\omega m e_0 = \omega' m e_0$. Since the condition holds, we have $\omega m e_1 = \omega' m e_1$ that is, $\omega = \omega'$. □

Theorem 7.2 *If $\Gamma \vdash A$ has a composition structure and $\Gamma.A \vdash B$ sset then $\Gamma \vdash \Pi\ A\ B$ sset.*

Proof. We use the previous Lemma 7.1. Assume that we have w and w' in $(\Pi\ A\ B)(I^+, \rho)$ such that $we_0 = we_1$. We prove $we_0 = w'e_0$. For this we take an arbitrary $f : J \to I$ and we prove $we_1 f\ u = w'e_1 f\ u$ in $B(J, \rho e_1 f, u)$ for any u in $A(J, \rho e_1 f)$. We have $e_1 f = f^+ e_1$ and since A has a composition structure, we can find $\tilde{u}$ ub $A(J^+, \rho f^+)$ such that $\tilde{u} e_1 = u$. Since B is a strict set and we have $we_0 f\ u = (wf^+\ \tilde{u})e_0 = (w'f^+\ \tilde{u})e_0 = w'e_0 f\ u$ we also have $we_1 f\ u = (wf^+\ \tilde{u})e_1 = (w'f^+\ \tilde{u})e_1 = w'e_1 f\ u$ as desired. □

8 The universe of strict propositions is a strict set

If $\Gamma \vdash A$ is a strict proposition each set $A(I, \rho)$ is a subsingleton. We can think of this set as a truth value, and write $A(I, \rho)$ *true* to mean that it is inhabited.

Lemma 8.1 *If $\Gamma \vdash A$ is a strict proposition that has a composition structure then for any ρ in $\Gamma(I^+)$, we have that $A(I^+, \rho)$ is true if, and only if, both $A(I, \rho e_0)$ and $A(I, \rho e_1)$ are true if, and only if one of $A(I, \rho e_0)$ or $A(I, \rho e_1)$ is true.*

While it does not seem possible to prove that sProp is a strict set in our purely abstract and axiomatic framework, it is possible to prove it in the concrete case of the cubical set model, where the objects of the base category are finite sets. In this case indeed, it follows from the Lemma that if $\Gamma \vdash A$ is a strict proposition that has a composition structure, then $A(I, \rho)$ is true as soon as there exists $f : \emptyset \to I$ such that $A(\emptyset, \rho f)$ is true. We get then the following result in this concrete case.

Corollary 8.2 *if $\Gamma \vdash A$ is a strict proposition that has a composition structure, and $f : J \to I$ and $\rho : I \to \Gamma$ then $A(J, \rho f)$ is true if, and only if, $A(I, \rho)$ is true.*

From this remark we get that if we have $I^+ \vdash A\ A'$ are strict propositions that have a composition structure, and $I \vdash Ae_0 = A'e_0$ then $I^+ \vdash A = A'$. Indeed, if $f : J \to I^+$ and $A(J, f)$ is true then so is $A(I^+, 1)$ and also $A(I, e_0) = A'(I, e_0)$ and then $A'(I^+, 1)$ is true and also $A'(J, f)$.

It follows that sProp is a strict set.

Bishop sets

Similarly, while it does not seem possible to prove that Bishop sets are closed by dependent product in our purely abstract and axiomatic framework, it is possible to prove it in the concrete case of the cubical set model, where the objects of the base category are finite sets. In this case we have the following alternative characterisation.

Proposition 8.3 *$\Gamma \vdash A$ is a Bishop sets if, and only if, any two element u, u' of $A(I, \rho)$ are equal as soon as we have $uf = u'f$ in $A(\emptyset, \rho f)$ for all $f : \emptyset \to I$.*

Corollary 8.4 *If $\Gamma.A \vdash B$ is a Bishop set then $\Gamma \vdash \Pi\ A\ B$ is a Bishop set.*

Proof. We take w and w' in $(\Pi\ A\ B)(I, \rho)$ and assume that we have $wf = w'f$ in $(\Pi\ A\ B)(\emptyset, \rho f)$ for all $f : \emptyset \to I$. If we take $g : J \to I$ and u in $A(J, \rho g)$ we show $wg\ u = w'g\ u$ in $B(J, \rho g, u)$. Indeed, since B is a Bishop set, it is enough to show $(wg\ u)h = (w'g\ u)h$ in $B(\emptyset, \rho gh, uh)$ for all $h : \emptyset \to J$, but we have

$$(wg\ u)h = wgh\ uh = w'gh\ uh = (w'g\ u)h$$

since $wgh = w'gh$ by hypothesis. □