REMARK ON THE FORSTER-SWAN THEOREM

1. A variation on the stable range theorem

In [1] we present a non Noetherian generalisation of Swan's theorem [4] on the number of generators of a module. It turns out that the main lemma and the proof can be slightly simplified, by stating a general corollary of the main lemma. Direct applications of this corollary are a refined version of Swan's theorem and Bass cancellation theorem. This simplification was found by email discussions with Lionel Ducos.

Let R be a commutative ring with unit. We follow the notation of [1], by writing $D(L)$, if $L \in R^n$, the radical ideal generated by L. Thus $D(L) = 1$ means that L is unimodular.

Lemma 1.1. *Assume given $C, C_1 \dots, C_n$ vectors in R^m, and $u_1, \dots, u_n \in R$. If $\mathsf{Hdim}(R) < n$ and*

$$1 = D(a) \vee D(C)$$

then there exist $x_1, \dots, x_n$, all multiple of a, such that

$$1 = D(u_1 + ax_1, \dots, u_n + ax_n) \vee D(C + x_1C_1 + \cdots + x_nC_n)$$

Corollary 1.2. *Assume $C, C_1, \dots, C_n$ are column vector in R^m and ν is a minor of order n of $C_1, \dots, C_n$. If $1 = D(\nu) \vee D(C)$ and $\mathsf{Hdim}(R) < n$ then there exists $x_1, \dots, x_n \in R$ such that $C + x_1C_1 + \cdots + x_nC_n$ is unimodular.*

The lemma is a special case of lemma 5.1 of [1] and can be proved directly by the same argument. The corollary follows by taking $a = \nu$ and u_i to be the minor obtained by replacing C_i by C in $C_1, \dots, C_n$. We notice then that $u_i + ax_i$ is also the minor obtained by replacing C_i by $C + x_1C_1 + \cdots + x_nC_n$ in $C_1, \dots, C_n$ so that

$$1 = D(u_1 + ax_1, \dots, u_n + ax_n) \vee D(C + x_1C_1 + \cdots + x_nC_n)$$

implies that $C + x_1C_1 + \cdots + x_nC_n$ is unimodular.

2. Applications

If M is a rectangular matrix, we let $\Delta_n(M)$ be $\bigvee_\nu D(\nu)$ where ν varies over all minors of M of order n. Like in [1] let now F be a rectangular matrix of elements in R of columns $C, C_1, \dots, C_p$, and G the matrix of columns $C_1, C_2, \dots, C_p$.

Corollary 2.1. *If $D(C) \vee \Delta_n(G) = 1$ and $\mathsf{Hdim}(R) < n$ then there exists $t_1, \dots, t_p$ such that $C + t_1C_1 + \cdots + t_pC_p$ is unimodular.*

Proof. From $D(C) \vee \Delta_n(G) = 1$, we get a family $\nu_1, \dots, \nu_p$ of minors of G of order n such that $D(C) \vee \bigvee_i D(\nu_i)$. We apply then successively corollary 1.2 to get $D(C) \vee D(\nu_k) = 1$ in R/J_k with $J_k = \bigvee_{i>k} D(\nu_i)$ until we get $D(C) = 1$ in R. $\square$

The refined version of Swan's theorem is the following.

Corollary 2.2. *Suppose that* $1 = \Delta_1$ *and that for each* $k > 0$ *and for each minor* ν *of* G *of order* n *we have* $\mathsf{Hdim}(R/\Delta_{n+1}) < n$. *Then there exist* $t_1, \ldots, t_p$ *such that the vector* $C_0 + t_1C_1 + \cdots + t_pC_p$ *is unimodular.*

Proof. We apply the corollary 2.1 successively in $R/\Delta_2,\ R/\Delta_3, \ldots$ □

Corollary 2.3. *If* $\Delta_n(G) = 1$ *and* $\mathsf{Hdim}(R) < n$ *and* $D(C) \vee D(a) = 1$ *then there exists* $t_1, \ldots, t_p$ *such that* $C + at_1C_1 + \cdots + at_pC_p$ *is unimodular.*

Proof. We apply the corollary 2.1 to the vectors $C, aC_1, \ldots, aC_n$. □

Theorem 2.4. *If* $\mathsf{Hdim}(R) < n$ *and* P, Q, M *are finitely generated projective module over* R *such that* P *is of rank* $\geq n$ *and* $P \oplus M \simeq Q \oplus M$ *then* $P \simeq Q$.

Proof. For completness we reproduce the argument from [2]. We reduce the statement to the case where $M = R$. We can also assume that P is the image in some R^k of an idempotent square matrix. We have an isomorphism $\phi : Q \oplus R \simeq P \oplus R$ and we let (C, a) be the vector $\phi(0, 1)$. Since (C, a) is unimodular, by using corollary 2.3, we have $C' \in P$ such that $C + aC'$ is unimodular. We then have a row vector X such that $XC + aXC' = 1$. The transformation $(V, x) \longmapsto (V + xC', -aXV + xXC)$ is then bijective and sends (C, a) to $(C + aC', 0)$. Since $C + aC'$ is unimodular there is a bijection that transforms also $(C + aC', 0)$ to $(0, 1)$. □

References

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