Some examples of complete Cisinski model structures

Introduction

We assume given a Grothendieck universe $\mathcal{U}$ and a category $\mathcal{C}$ in this universe. We write $\hat{\mathcal{C}}$ the category of presheaves on $\mathcal{C}$ and denote by $Yo : \mathcal{C} \to \hat{\mathcal{C}}$ the Yoneda functor. The goal of this note is to describe a new class of *complete* Cisinski model structures (abreviated as "C-model structures"). These are the C-model structures determined by a segment $\mathbb{I}$, i.e. a presheaf with two distinct elements 0 and 1, satisfying the following two conditions

1. $\mathbb{I}$ has a *connection structure*, i.e. two maps $\wedge, \vee : \mathbb{I}^2 \to \mathbb{I}$ such that $x \wedge 0 = 0 \wedge x = 0,\ x \wedge 1 = 1 \wedge x = x$ and $x \vee 0 = 0 \vee x = x,\ x \vee 1 = 1 \vee x = 1$
2. there is a functor $J \longmapsto J^+$ on $\mathcal{C}$ with a natural isomorphism $Yo(J^+) \simeq \mathbb{I} \times Yo(J)$

The first condition holds for the "Lawvere-segment" (subobject classifier) Ω, but the second condition will then typically not hold. An example where the two conditions are realized is when $\mathcal{C}$ is the Lawvere theory associated to the equational theory of distributive lattices (resp. Boolean algebra) and $\mathbb{I}$ the generic distributive lattice (resp. Boolean algebra). Another example is the category of nonempty finite sets and arbitrary maps, with $\mathbb{I}$ being represented by the set $\{0, 1\}$.

For proving that the associated model structure is complete, we are going to build another model structure ("S-model structure") where the fibrations are the *naive* fibrations and cofibrations are the monomorphisms. This S-model structure will then have the same fibrant objects and cofibrations and hence will coincide with the C-model structure. It will follow that the two notions of naive fibrations and (Cisinski) fibrations coincide.

One key point will be to build a (naive) $\mathcal{U}$-fibration $\tilde{\mathsf{F}} \to \mathsf{F}$, with F fibrant, which is universal for the $\mathcal{U}$-fibrations, i.e. for any (naive) $\mathcal{U}$-fibration $X \to Y$ we have a (non necessarily unique) pull-back square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \tilde{\mathsf{F}} \arrow[d] \\ Y \arrow[r] & \mathsf{F} \end{tikzcd}$$

1 Cofibrations, anodyne maps, fibrations and trivial fibrations of the S-model structure

We write $I, J, K, \ldots$ the objects of the given $\mathcal{U}$-category $\mathcal{C}$. We write $\Gamma, \Delta, X, Y, , \ldots$ for presheaves over $\mathcal{C}$ and T is the terminal presheaf. A presheaf X is given by a collection of sets $X(I)$ with restriction maps $X(I) \to X(J)$ sending u to uf for $f : J \to I$ satisfying the usual functorial equations. We write Ω the subobject classifier: $\Omega(I)$ is the set of sieves on I.

Any map $\psi : X \to \Omega$ defines a subpresheaf $X|\psi$ of X where $(X|\psi)(I)$ is the subset of element ρ in $X(I)$ such that $\psi\rho = 1$ in $\Omega(I)$. We write $\iota_\psi : X|\psi \to X$ the canonical inclusion. If $\sigma : Y \to X$ and $\psi : X \to \Omega$ we denote by $\psi^*\sigma : Y|\psi\sigma \to X|\sigma$ the map $(\psi^*\sigma)(\nu) = \sigma(\nu)$. If we have $\psi\sigma = 1$ in $Y \to \Omega$ then σ factorizes to a map $Y \to X|\psi$ that we still write σ.

We say that a map is a *cofibration* if, and only if, it is a monomorphism. A *trivial fibration* is a map that has the right lifting property w.r.t. any monomorphism. In particular the map $X \to T$ is a trivial fibration if, and only if, X is an injective object.

A (generalized) open box is a map of the form $A \times \mathbb{I} \cup B \times 0 \to B \times \mathbb{I}$ or $A \times \mathbb{I} \cup B \times 1 \to B \times \mathbb{I}$ for $A \to B$ monomorphism. A *fibration* is a map that has the right lifting property w.r.t. any open box.

Finally a map is *anodyne* if it has the left lifting property w.r.t. any fibration.

We are going to prove that the *pullback of an andoyne map along any fibration is anodyne* (using only condition 1 on the segment) and that *we can extend a fibration along any anodyne map* (using both conditions).

We indicate then how to define weak equivalence and conclude the proof that these notions of cofibrations, fibrations and weak equivalence do define a model structure.

2 Types and elements

From now on, we will identify $\mathcal{C}$ with the full subcategory of $\hat{\mathcal{C}}$ via the Yoneda embedding. We have a natural bijection $I \to \Gamma \simeq \Gamma(I)$: if $\rho : I \to \Gamma = Yo(I) \to \Gamma$ then $\rho(1_I)$ is an element of $\Gamma(I)$ and if u is in $\Gamma(I)$ then we can define $\overline{u} : I \to \Gamma$ by $\overline{u}(f) = uf$, in such a way that $\overline{\rho(1)} = \rho$ and $\overline{u}(1) = u$.

Given a presheaf Γ, the category of elements of Γ is the category $Yo \downarrow \Gamma$, so that an object of this category is of the form (I, ρ) with $\rho : I \to \Gamma$ and a map $f : (J, \nu) \to (I, \rho)$ is a map $f : J \to I$ such that $\rho f = \nu$.

We write $\mathsf{Type}(\Gamma)$ the set of all $\mathcal{U}$-valued presheaf on the category of elements of Γ. We let $\mathsf{Elem}(\Gamma, A)$ be the set of global sections of A: it is given by a family $u(I, \rho)$ in $A(I, \rho)$ such that $u(I, \rho)f = u(J, \rho f)$ if $f : J \to I$. If $\sigma : \Delta \to \Gamma$ and A in $\mathsf{Type}(\Gamma)$ we define by composition an element $A\sigma$ of $\mathsf{Type}(\Delta)$. If u is in $\mathsf{Elem}(\Gamma, A)$ we define $u\sigma$ in $\mathsf{Elem}(\Delta, A\sigma)$ by taking $u\sigma(I, \nu) = w(I, \sigma(\nu))$.

If A is in $\mathsf{Type}(\Gamma)$ we define a new presheaf $\Gamma.A$ by taking $(\Gamma.A)(I)$ to be the set of pairs ρ, u with ρ in $\Gamma(I)$ and u in $A(I, \rho)$. We define a projection $p_A : X.A \to X$ by $p_A(\rho, u) = \rho$ and an element q_A in $\mathsf{Elem}(X.A, Ap_A)$ by $q_A(I, (\rho, u)) = u$. We may write simply p and q instead of p_A and q_A if A is clear from the context.

If $\sigma : \Delta \to \Gamma$ and A is in $\mathsf{Type}(\Gamma)$ and u is in $\mathsf{Elem}(\Delta, A\sigma)$, we can define $(\sigma, u) : \Delta \to \Gamma.A$ by $(\sigma, u)(\nu) = \sigma(\nu), u(I, \sigma(\nu))$ for ν in $\Delta(I)$. We then have $p(\sigma, u) = \sigma$ and $q(\sigma, u) = u$. If a is in $\mathsf{Elem}(\Gamma, A)$ we write $[a] = (1, a) : \Gamma \to \Gamma.A$. In this way, if we have B in $\mathsf{Type}(\Gamma.A)$ then $B[a]$ is in $\mathsf{Type}(\Gamma)$.

From the hypotheses on the segment we deduce natural transformations $e_0, e_1 : I \to I^+$ and $p : I^+ \to I$ such that $pe_0 = pe_1 = 1_I$ We also have the maps $\delta_0, \delta_1 : I^+ \to \Omega$ corresponding to the monomorphisms e_0 and $e_1 : I \to I^+$.

The following remark will be useful. Given A in $\mathsf{Type}(\Gamma)$ an element a in $\mathsf{Elem}(\Gamma, A)$ is completely determined by a family of elements u_ρ in $\mathsf{Elem}(I, A\rho)$ for $\rho : I \to \Gamma$ such that $u_\rho f = u_{\rho f}$ for $f : J \to I$. We say then that we define such an element by the equations $a\rho = u_\rho$. Using the natural isomorphism $J^+ \simeq J \times \mathbb{I}$ we can refine this remark to the following result.

Lemma 2.1 *If $\psi : I \to \Omega$ and A in $\mathsf{Type}(I^+|\psi p)$, an element in $\mathsf{Elem}(I^+|\psi p, A)$ is completely determined by a family u_f in $\mathsf{Elem}(J^+, Af^+)$, for $f : J \to I$ such that $\psi f = 1$, satisfying $u_f g^+ = u_{fg}$ if $g : K \to J$.*

3 Partial elements and trivial fibration structures

If A is in $\mathsf{Type}(\Gamma)$ and $\rho : I \to \Gamma$, a *partial element* of A at ρ is given by a pair $\psi \mapsto a$ with $\psi : I \to \Omega$ and a in $\mathsf{Elem}(I|\psi, A\rho\iota_\psi)$. We say that ψ is the *extent* of this partial element. The partial element is *total* if $\psi = 1$. If $u = \psi \mapsto a$ is a partial element at ρ and $f : J \to I$ we define $uf = \psi f \mapsto a\psi^*(f)$ partial element at $\rho f : J \to \Gamma$. If $\varphi \leqslant \psi$ we have a canonical map $\iota : I|\varphi \to I|\psi$ and we write $u|\varphi$, partial element of extent φ, instead of $u\iota$. In general a partial element u of extent φ is *compatible* with a partial element v of extent ψ if $u|\varphi \wedge \psi = v|\varphi \wedge \psi$. We can then form the join $u \vee v$ of u and v (of extent $\varphi \vee \psi$) in a natural way. *The following notation will be convenient*: the join of two compatible partial elements $\varphi \mapsto a$ and $\psi \mapsto b$ will be written $\varphi \mapsto a, \psi \mapsto b$.

A *trivial fibration structure* κ on A is an operation which extends a partial element to a total element, that is $\kappa(\rho, u)$ is in $\mathsf{Elem}(I, A)$ such that $\kappa(\rho, u)\iota_\psi = u$ if u is of extent ψ and which satisfies the uniformity condition $\kappa(\rho, u)f = \kappa(\rho f, uf)$ if $f : J \to I$.

We have the following remark, which is a direct generalisation of the fact that an object X is injective if, and only if, it has a uniform extension operation given by a left inverse of the inclusion of X in its object of subsingleton.

Theorem 3.1 *Given A in $\mathsf{Type}(\Gamma)$, the projection $p : \Gamma.A \to \Gamma$ is a trivial fibration if, and only if, A has a trivial fibration structure.*

4 Open boxes, filling and composition structures

We use this to define the notion of lower and upper open boxes for A in $\mathsf{Type}(\Gamma)$ at $\rho : I^+ \to \Gamma$. A *lower (resp. upper) open box* is given by a partial element b of extent $\delta_0 \vee \psi p$ (resp. $\delta_1 \vee \psi p$) for some $\psi : I \to \Omega$. Since[1] $\psi = (\delta_0 \vee \psi p)e_1$, we can recover $\psi : I \to \Omega$ from the element b. A *missing lid* for such a lower open box $b = (\delta_0 \mapsto a_0, \psi p \mapsto a)$ is an element a_1 in $\mathsf{Elem}(I, A\rho e_1)$ such that $a_1\iota_\psi = a(\psi p)^* e_1$ (and similarly for upper open box).

A *filling structure* (resp. composition structure) for A in $\mathsf{Type}(\Gamma)$ is given by an operation $\kappa(\rho, b)$ which takes a lower open box and produces a filler (resp. missing lid) for this box, and which furthermore satisfies the following uniformity condition that we have $\kappa(\rho, b)f^+ = \kappa(\rho f^+, bf^+)$ (resp. $\kappa(\rho, b)f = \kappa(\rho f^+, bf^+)$), if $f : J \to I$, together with a similar operation on upper open boxes. We write $\mathsf{Fill}(\Gamma, A)$ (resp. $\mathsf{Comp}(\Gamma, A)$) the set of all such filling (resp. composition) structures.

If κ is in $\mathsf{Fill}(\Gamma, A)$ and $\sigma : \Delta \to \Gamma$ we define $\kappa\sigma$ in $\mathsf{Fill}(\Delta, A\sigma)$ by the equation $\kappa\sigma(\nu, b) = \kappa(\sigma\nu, b)$ and similarly for κ in $\mathsf{Comp}(\Gamma, A)$.

As for trivial fibration structures, we have the following result.

Theorem 4.1 *Given a type A in $\mathsf{Type}(\Gamma)$, the projection $p : \Gamma.A \to \Gamma$ is a fibration if, and only if, A has a filling structure.*

We can now use the connection structure on $\mathbb{I}$ to reduce the notion of *filling* structure to the notion of *composition* structure. To the connections $\wedge, \vee$ on $\mathbb{I}$ correspond operations $\mu, \delta : I^{++} \to I^+$ respectively. We thus have for instance $\mu e_1 = \mu e_1^+ = 1$ and $\mu e_0 = \mu e_0^+ = e_0 p$.

Lemma 4.2 *We define an operation that takes a lower open box b at $\rho : I^+ \to \Gamma$ and produces an open box $L(b)$ at $\rho\mu : I^{++} \to \Gamma$ and such that $L(b)e_1 = b$. Furthermore $L(b)f^{++} = L(bf^+)$ if $f : J \to I$.*

Proof. We define $L(\delta_0 \mapsto a_0, \psi p \mapsto a) = (\delta_0 \mapsto a_0\delta_0^*(p^+), \delta_0 p \mapsto a_0\delta_0^* p, \psi pp \mapsto a(\psi p)^*\mu)$. □

Theorem 4.3 *The set $\mathsf{Comp}(\Gamma, A)$ is a retract of the set $\mathsf{Fill}(\Gamma, A)$ (in a natural way).*

Proof. If κ_A is a filling structure, it is clear that we define a (lower) composition structure c_A by the equation $c_A(\rho, b) = \kappa_A(\rho, b)e_1$.

Conversely, given a composition structure c_A we define a filling structure κ_A by the equation $\kappa_A(\rho, b) = c_A(\rho\mu, L(b))$. This defines a filling structure which satisfies $\kappa_A(\rho, b)e_1 = c_A(\rho, b)$. □

5 Dependent products

Given B in $\mathsf{Type}(\Gamma.A)$ we define $\Pi(A, B)$ in $\mathsf{Type}(\Gamma)$ by taking $\Pi(A, B)(I, \rho)$ to be the set of families $\lambda_f,\ f : J \to I$ such that λ_f is a function taking an element u in $\mathsf{Elem}(J, A\rho f)$ and producing an element $\lambda_f(u)$ is in $\mathsf{Elem}(J, B(\rho f, u))$, and we have $(\lambda_f(u))g = \lambda_{fg}(ug)$ if $g : K \to J$. If $\sigma : \Delta \to \Gamma$ we then have $\Pi(A, B)\sigma = \Pi(A\sigma, B(\sigma p, q))$.

[1]This uses that 0 and 1 are distinct element of $\mathbb{I}$.

If c is in $\mathsf{Elem}(\Gamma, \Pi(A,B))$ and a is in $\mathsf{Elem}(\Gamma, A)$ we define $\mathsf{app}(c,a)$ in $\mathsf{Elem}(\Gamma, B[a])$ by the equation $\mathsf{app}(c,a)\rho = c\rho 1_I(a\rho)$. (We recall that $[a] = (1,a) : \Gamma \to \Gamma.A$.) If we have partial elements $v = \psi \mapsto c$ for $\Pi\ A\ B$ and $u = \psi \mapsto a$ for A at $\rho : I \to \Gamma$ of the same extent $\psi : I \to \Omega$ we also write $\mathsf{app}(v,u) = \psi \mapsto \mathsf{app}(c,a)$.

Using in a crucial way the reduction from filling to composition structures (and hence the connection structure of $\mathbb{I}$), we can show that dependent product can be lifted to filling structures.

Theorem 5.1 *Given filling structures κ_A on A and κ_B on B we can build a fibration structure $\pi(\kappa_A, \kappa_B)$ on $\Pi(A,B)$. Furthermore $\pi(\kappa_A,\kappa_B)\sigma = \pi(\kappa_A\sigma, \kappa_B(\sigma p, q))$ if $\sigma : \Delta \to \Gamma$.*

Proof. Using Theorem 4.3, we instead define a composition structure for $\Pi(A,B)$. Given $\rho : I^+ \to \Gamma$ and an open box b, of extent φ, at ρ, we explain how to build a missing lid λ in $\mathsf{Elem}(I, (\Pi\ A\ B)\rho e_1)$. Given $f : J \to I$ and a_1 in $\mathsf{Elem}(I, A\rho e_1 f)$ we define $\lambda_f(a_1)$ to be $\kappa_B(\rho f, \mathsf{app}(bf, a\iota_{\varphi f}))e_1$ where $a = \kappa_A(\rho f, \delta_1 \mapsto a_1 p \iota_{\delta_1})$. □

Corollary 5.2 *The pullback of an anodyne map along any fibration is anodyne.*

Proof. This follows from Theorems 4.1 and 5.1. □

6 Universe

So far, we only have used the connection structure on the segment $\mathbb{I}$ and could have used everywhere $J \times \mathbb{I}$ instead of J^+. The second hypothesis on the segment, that we can represent $J \times \mathbb{I}$ by J^+, will be used in a crucial way for the definition of the universe.

6.1 Definition of the universe

We define $\mathsf{Fib}(\Gamma)$ to be the set of pairs A, κ where A is in $\mathsf{Type}(\Gamma)$ and κ in $\mathsf{Fill}(\Gamma, A)$. If $X = (A, \kappa)$ is an element in $\mathsf{Fib}(\Gamma)$ we write $X.1 = A$ in $\mathsf{Type}(\Gamma)$ the first component of X.

The *universe* is the presheaf $\mathsf{F}(I) = \mathsf{Fib}(I)$. We define El in $\mathsf{Type}(\mathsf{F})$ by $\mathsf{El}(I, (A,\kappa)) = A(I, 1_I)$. With this definition, if $\rho : I \to \mathsf{F} = Yo(I) \to \mathsf{F}$ and $\rho(1) = (A, \kappa)$ we have $\mathsf{El}\rho = A$ since if $f : J \to I$ we have $\mathsf{El}\rho(J,f) = \mathsf{El}(J, \rho(f)) = \mathsf{El}(J, \rho(1)f) = \mathsf{El}(J, (Af, \kappa f)) = Af(J, 1_J) = A(J,f)$. Furthermore a partial element $\psi \mapsto u$ of El at $\rho : I \to F$ is an element in $\mathsf{Elem}(I|\psi, \mathsf{El}\rho\iota_\psi) = \mathsf{Elem}(I|\psi, A\iota_\psi)$ and thus it is the same as a partial element of A at 1_I.

Theorem 6.1 *We can build κ_E in $\mathsf{Fill}(\mathsf{F}, \mathsf{El})$ such that El, κ_E is "universal": if A, κ is in $\mathsf{Fib}(\Gamma)$ there exists a unique map $|(A,\kappa)| : \Gamma \to \mathsf{F}$ such that $\mathsf{El}|(A,\kappa)| = A$ and $\kappa_E|(A,\kappa)| = \kappa$.*

Proof. Given $\rho : I^+ \to \mathsf{F} = Yo(I^+) \to \mathsf{F}$ we can consider $\rho(1) = (A, \kappa)$ in $\mathsf{F}(I^+)$ and define $\kappa_E(\rho, b) = \kappa(1_{I^+}, b)$ which is in $\mathsf{Elem}(I^+, A) = \mathsf{Elem}(I^+, El\rho)$ using the fact that a box of El at ρ is the same as a box of A at 1_{I^+}.

If A is in $\mathsf{Type}(\Gamma)$ and κ is in $\mathsf{Fill}(\Gamma, A)$ we define $X : \Gamma \to \mathsf{F}$ by taking $X(\alpha) = A\overline{\alpha}, \kappa\overline{\alpha}$ where $\overline{\alpha} : I \to \Gamma$ is defined by $\overline{\alpha}(f) = \alpha f$ for $f : J \to I$. We then have $\mathsf{El}X = A$ since $\mathsf{El}X(I,\rho) = \mathsf{El}(I, X\rho) = A\rho(I, 1_I) = A(I,\rho)$ since $X\rho(1) = A\rho, \kappa\rho$ and $\kappa_E X = \kappa$ since $\kappa_E X(\rho, b) = \kappa_E(X\rho, b) = \kappa\rho(1_{I^+}, b) = \kappa(\rho, b)$ since $X\rho(1) = A\rho, \kappa\rho$. □

The rest of this section will be the proof that F is fibrant.

6.2 Alignment operation

Let ψ be a map $\Gamma \to \Omega$. The following result almost allows us to think of a fibration structure as a property. We use the fact that if $\varphi : I^+ \to \Omega$ then we can define $\forall\varphi : I \to \Omega$ such that $(\forall\varphi)f = 1$ if, and only if, $\varphi f^+ = 1$ for $f : J \to I$.

The following Lemma has a more conceptual proof by building first an alignment operation for *trivial* fibration structures, which is direct, and then reducing the case of fibration structures to the notion of trivial fibration structures using the notion of Leibitz exponential. It is then clear that what is needed if that $m^{\mathbb{I}} : A^{\mathbb{I}} \to B^{\mathbb{I}}$ is a cofibration if m is a cofibration, which corresponds to the use of the $\forall$ operation.

Lemma 6.2 *We can define an operation which, given* κ *is in* $\mathsf{Fill}(\Gamma, A)$ *and* $\psi : \Gamma \to \Omega$ *and* κ' *in* $\mathsf{Fill}(\Gamma|\psi, A\iota_\psi)$ *produces* $\mathsf{Al}(\kappa, \psi, \kappa')$ *in* $\mathsf{Fill}(\Gamma, A)$ *such that* $\mathsf{Al}(\kappa, \psi, \kappa')\iota_\psi = \kappa'$ *and furthermore satisfies* $\mathsf{Al}(\kappa, \psi, \kappa')\sigma = \mathsf{Al}(\kappa\sigma, \psi\sigma, \kappa'\psi^*(\sigma))$ *if* $\sigma : \Delta \to \Gamma$.

Proof. Given $\rho : I^+ \to \Gamma$ and b a box at ρ for A we define $\mathsf{Al}(\kappa, \kappa')(\rho, b)$ to be $\kappa(\rho, b \vee (\tau p \mapsto u))$ with $\tau = \forall(\psi\rho)$ in $I \mapsto \Omega$ and u in $\mathsf{Elem}(I^+|\tau p, A\rho\iota_{\tau p})$ is determined using Lemma 2.1: for $f : J \to I$ such that $\tau f = 1$, i.e. $\psi\rho f^+ = 1$ we define u_f in $\mathsf{Elem}(J^+, A\rho f^+)$ to be $\kappa'(\rho f^+, bf^+)$ so that $uf^+ = u_f$ $\square$

6.3 Equivalence extension property

If A and B are in $\mathsf{Type}(\Delta)$ we write $w : B \to A$ to mean that w is a natural transformation between B and A (that are presheaves on the category of elements of Δ). If $\sigma : \Delta_1 \to \Delta$ we define $w|\sigma : B\sigma \to A\sigma$ by $w|\sigma(v) = w(v)$ for v in $B\sigma(I, \nu) = B(I, \sigma\nu)$. If b is in $\mathsf{Elem}(\Delta_1, B\sigma)$ we define $\mathsf{app}(w, b)$ in $\mathsf{Elem}(\Delta_1, A\sigma)$ by $\mathsf{app}(w, b)(I, \nu) = w(b(I, \sigma\nu))$.

We define the set $\mathsf{Ext}(\Gamma)$ of *extension problems* at Γ as the set of all tuples $E = (A, \psi, B, w)$ with A in $\mathsf{Type}(\Gamma)$ and $\psi : \Gamma \to \Omega$ and B in $\mathsf{Type}(\Gamma|\psi)$ with $w : B \to A\iota_\psi$. If $\sigma : \Delta \to \Gamma$ we define $E\sigma = (A\sigma, \psi\sigma, B\psi^*\sigma, w|\sigma)$.

Lemma 6.3 *We can build operations* $\mathsf{G}(E)$ *in* $\mathsf{Type}(\Gamma)$ *and* $\mathsf{e}(E) : \mathsf{G}(E) \to A$ *such that* $\mathsf{G}(E)\iota_\psi = B$ *and* $\mathsf{e}(E)|\iota_\psi = w$. *Furthermore* $\mathsf{G}(E)\sigma = \mathsf{G}(E\sigma)$ *and* $\mathsf{e}(E)|\sigma = \mathsf{e}(E\sigma)$ *if* $\sigma : \Delta \to \Gamma$.

Proof. We define the set $\mathsf{G}(E)(I, \rho)$ for $\rho : I \to \Gamma$ by case on $\psi\rho : I \to \Omega$. If $\psi\rho = 1$ we take it to be the set $B(I, \rho)$ since then ρ can be seen as a map $I \to \Gamma|\psi$, and if $\psi\rho \neq 1$ we take it to be the set of pairs a, b with a in $\mathsf{Elem}(I, A\rho)$ and b in $\mathsf{Elem}(I|\psi\rho, A\psi^*(\rho))$ such that $a\iota_\psi = \mathsf{app}(w, b)$. We define then $\mathsf{e}(E)(b) = w(b)$ if b is in $\mathsf{G}(E)(I, \rho) = B(I, \rho)$ if $\psi\rho = 1$ and $\mathsf{e}(E)(a, b) = a(I, 1_I)$ if $\psi\rho \neq 1$. $\square$

Given a in $\mathsf{Elem}(\Gamma, A)$ and b in $\mathsf{Elem}(\Gamma|\psi, B)$ such that $\mathsf{app}(w, b) = a$ we define $\mathsf{glue}(a, b)$ in $\mathsf{Elem}(\Gamma, \mathsf{G}(E))$ by taking $\mathsf{glue}(a, b)(I, \rho) = b(I, \rho)$ if $\psi\rho = 1$ and $\mathsf{glue}(a, b)(I, \rho) = (a\rho, b\psi^*(\rho))$ if $\psi\rho \neq 1$.

If A and B are in $\mathsf{Type}(\Delta)$ an *equivalence structure* for a map $w : B \to A$ is given by an operation ϵ which express that all fibers of w are "contractible". It takes $\rho : I \to \Delta$ and a in $\mathsf{Elem}(I, A\rho)$ and a partial element $\psi \mapsto b$ of B at ρ such that $\mathsf{app}(w, b) = a\iota_\psi$ and produces a pair $(\tilde{b}, \omega) = \epsilon(\rho, a, \psi \mapsto b)$ where $\tilde{b}$ in $\mathsf{Elem}(I, B\rho)$ is such that $\tilde{b}\iota_\psi = b$ and ω in $\mathsf{Elem}(I^+, A\rho p)$ is such that $\omega e_1 = a$ and $\omega e_0 = \mathsf{app}(w, \tilde{b})$ and $\omega\iota_{\psi p} = ap\iota_{\psi p}$ (i.e. ω is constant on the extend ψ). Furthermore, if $f : J \to I$ we should have $\epsilon(\rho f, af, vf) = (\tilde{b}f, \omega f^+)$.

A *filling structure* for $E = (A, \psi, B, w)$ is a tuple $\kappa_E = (\kappa_A, \psi, \kappa_B, \epsilon_w)$ where κ_A is in $\mathsf{Fill}(\Gamma, A)$ and κ_B in $\mathsf{Fill}(\Gamma|\psi, B)$ and ϵ_w is an equivalence structure for w. We define $\kappa_E\sigma = (\kappa_A\sigma, \psi\sigma, \kappa_B\psi^*\sigma, \epsilon_w\psi^*\sigma)$ if $\sigma : \Delta \to \Gamma$.

Lemma 6.4 *We can lift the previous operation* G *to filling structures: we can build* $\mathsf{g}(\kappa_E)$ *in* $\mathsf{Fill}(\Gamma, \mathsf{G}(E))$ *such that* $\mathsf{g}(\kappa_E)\sigma = \mathsf{g}(\kappa_E\sigma)$ *if* $\sigma : \Delta \to \Gamma$ *and* $\mathsf{g}(\kappa_E)\iota_\psi = \kappa_B$.

Proof. Using Lemma 6.2 we can forget the last condition. The proof is then similar to the proof of the extension of an equivalence along a cofibration in the simplicial set model. We could follow this proof, but instead describe here explicitely a filling structure κ for $\mathsf{G}(E)$.

Let v be an open box for G at $\rho : I^+ \to \Gamma$. We can form the open box $\mathsf{app}(e, v)$ for A at ρ and define $a' = \kappa_A(\rho, \mathsf{app}(\mathsf{e}(E)e, v))$. We define then $(b, \omega) = \epsilon_w(a', \psi \wedge \varphi, v)$ where φ is the extent of v. If $a = \kappa_A(\rho p, (\delta_1 \mapsto a'pp, \psi p \mapsto \omega))$ then we can take $\kappa(\rho, v) = \mathsf{glue}(a, b)$. $\square$

With these hypotheses, We can also show that the extension $\mathsf{e}(E) : \mathsf{G}(E) \to A$ of the map w also has an equivalence sructure, but we will not this fact here. (It corresponds to the fact that the universe we build is *univalent*.)

6.4 The universe is fibrant

We denote by $[i] : \Gamma \to \Gamma \times \mathbb{I}$ the map $x \mapsto (x, i)$ for $i = 0, 1$.

Lemma 6.5 *From any E is in $\mathsf{Fib}(\Gamma \times \mathbb{I})$ we can build a map $e(E) : E.1[0] \to E.1[1]$ with an equivalence structure $\epsilon(E)$. Furthermore, if $\sigma : \Delta \to \Gamma$ we have $e(E)|\sigma = e(E(\sigma \times \mathbb{I}))$ and $\epsilon(E)\sigma = \epsilon(E(\sigma \times \mathbb{I}))$.*

Lemma 6.6 *We can build an operation κ which takes A in $\mathsf{Fib}(\Gamma)$ and $\psi : \Gamma \to \Omega$ and E in $\mathsf{Fib}((\Gamma|\psi) \times \mathbb{I})$ and produces $\kappa(A, \psi, E)$ in $\mathsf{Fib}(\Gamma)$ such that $\kappa(A, \psi, E)\iota_\psi = E[0]$ and $\kappa(A, \psi, E)\sigma = \kappa(A\sigma, \psi\sigma, E(\sigma \times \mathbb{I}))$.*

Proof. This follows from Lemma 6.5 and the equivalence extension Lemma 6.4. □

Corollary 6.7 *The map $\mathsf{F} \to T$ is a fibration.*

Proof. Lemma 6.6 (and its dual swapping 0 and 1) defines essentially a composition structure for F. We can then conclude by Theorems 4.3 and 4.1. □

Corollary 6.8 *We can extend fibrations along anodyne maps.*

Proof. We assume given $\sigma : \Delta \to \Gamma$ which is anodyne, and B in $\mathsf{Type}(\Delta)$ such that $p_B : \Delta.B \to \Delta$ which is a fibration. By Theorem 4.1, B has a fibration structure (which may not be uniquely determined) κ_B. By Theorem 6.1, we have a map $Y = |(B, \kappa_B)| : \Delta \to \mathsf{F}$ such that $\mathsf{El}Y = B$ and $\kappa_E Y = \kappa_B$. Since F is fibrant and σ is anodyne, we can find a map $X : \Gamma \to \mathsf{F}$ such that $X\sigma = Y$. The element $A = \mathsf{El}X$ is then such that $A\sigma = (\mathsf{El}X)\sigma = \mathsf{El}(X\sigma) = \mathsf{El}Y = B$ and $\kappa_E X$ is a fibration structure for A. (Note that this fibration structure κ_A satisfies furthermore $\kappa_A\sigma = \kappa_B$ but we don't need this fact.) □

7 S-model structure

It can be shown that any map $\alpha : X \to Y$ can be written $\alpha = pi$ where i is a cofibration and p a trivial fibration and as $\alpha = qj$ where j is a trivial fibration and q a fibration. More precisely, we can build an element A in $\mathsf{Type}(Y)$ with a trivial fibration structure and a map $i : X \to Y.A$ which is a cofibration and such that $p_A i = \alpha$ and we can build an element B in $\mathsf{Type}(Y)$ with a fibration structure and a map $j : X \to Y.A$ which is anodyne and such that $p_B j = \alpha$. The first decomposition is direct, while the second decomposition requires an (effective) inductive argument

We can then define α to be a *weak equivalence* either by the fact that p_B is a *trivial* fibration (definition 1) or by that fact that i is anodyne (definition 2).

The surprising result is that, with either of these definitions, a map is a trivial fibration (resp. anodyne) if, and only if it is a fibration (resp. cofibration) and a weak equivalence.

Here is the structure of the remaining of Sattler's argument. What we cover in this paper is essentially the point 4:

1. We have 2-out-of-3 for trivial fibrations among fibrations.
2. For any triangle $A \to B \to C$ with $A \to B$ anodyne, fibration $B \to C$, and cofibration $A \to C$, we have that $B \to C$ is a trivial fibration exactly if $A \to C$ is anodyne.
3. Let (j_1, p_1) and (j_2, p_2) be two factorizations of a map into an anodyne map followed by a fibration. If p_1 is a trivial fibration, then so is p_2.
4. We can extend fibrations along anodyne maps, trivial fibrations along cofibrations, and trivial fibrations along anodyne maps. (This property allows us to compose factorizations.)
5. We have 2-out-of-3 for weak equivalences (as per definition 1).
6. Definitions 1 and 2 of weak equivalences are equivalent.
7. A cofibration is anodyne exactly if it is a weak equivalence.
8. A fibration is a trivial fibration exactly if it is an equivalence.