Variation on Cubical sets

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Introduction

In the model presented in [1, 4] a type is interpreted by a nominal set A equipped with two "face" operations: if $u : A$ and i is a symbol we can form $u(i = 0) : A(i = 0)$ and $u(i = 1) : A(i = 1)$ elements independent of i. The unit interval is represented by the nominal set $\mathbf{I}$, whose elements are $0, 1$ and the symbols. Alternatively, we have a presentation as a presheaf category over the category where objects are finite sets of symbols and morphisms $I \to J$ are determined by a disjoint union $I = I_0, I_1, I'$ and an injection $I' \to J$, and the presheaf $\mathbf{I}$ is then $\mathbf{I}(J) = J + \{0, 1\}$. The presheaf $\mathbf{I}$ does not satisfy the Kan filling condition. The equivalence between the two presentations is shown in [4].

In this model, if A represents a type, the path space of A is represented by the affine exponential $\mathbf{I} \to *A$, which is adjoint to the separated product $B * \mathbf{I}$ with $(b, i) \in B * \mathbf{I}$ if i is independent of b.

We modify this model by adding the operations $i \wedge j$ and $i \vee j$ in the set $\mathbf{I}$. This corresponds to adding *connections* [2]. We also have negations, and no linearity conditions. The set $\mathbf{I}$ becomes then the free de Morgan algebra on the set of symbols. Alternatively a morphism on the base category $I \to J$ is now a map $I \to \mathsf{dM}(J)$ from the set I to the free de Morgan algebra over the set J.

Using connections we can reduce the Kan filling operation to the simpler composition operations and we can interpret more judgemental equalities. In particular the computation of the elimination rule for equality is interpreted in a judgemental way.

One important point is that we cannot hope this computation rule to be interpreted as a *definitional* equality, since reflexivity is not considered to be an introduction rule/constructor anymore in the present approach. Instead, we interpret dependent types as *regular* fibrations (the lifting of a constant path is constant) and we check that this regularity condition is preserved by all type formers.

1 General remarks about cubical sets

All the notions of cubes are described as (covariant) presheaf categories over a category where the objects are finite set of symbols $I, J, K, \ldots$

Each time there is a direct geometric interpretation: if X is a topological space, then X defines a presheaf over the base category by taking $X(I)$ to be the set of continuous maps $[0, 1]^I \to X$.

For the simplest notion of cubes the morphisms $I \to J$ are determined by a disjoint decomposition $I = I_0, I_1, I'$ and an injection $I' \to J$ (the element of I_0 are sent to 0 element of I_1 to 1, and the symbols in I' are renamed). For instance if $I = i, j, k$ and $J = j$ and we define $if = 0,\ jf = j,\ kf = 1$ then the map $f : I \to J$ sends a cube $u(i, j, k)$ to the line $u(0, j, 1)$ (where intuitively, i, j, k vary in $[0, 1]$).

This is the category considered in the paper [1].

There we don't have a diagonal operations. Adding the diagonal operation corresponds to allowing any map $I' \to J$. For instance if $I = i, j$ and $J = k$ and we define $f : I \to J$ by $if = jf = k$, then the map f corresponds to sending a square $u(i, j)$ to its diagonal $u(k, k)$.

(The category we obtain in this way should be equivalent to what Grothendieck calls the "simplest test category" in [3].)

To add connections corresponds to adding the operations $i \wedge j$ and $i \vee j$ on symbols. For instance, if $I = i$ and $J = j, k$ the map $if = j \wedge k$ corresponds to sending the line $u(i)$ to the square $u(j \wedge k)$. The faces of this square are two constant lines $u(0)$ and two copies of the line $u(i)$.

If we have connections and not diagonal, we can now describe a morphism $I \to J$ as a map $I \to D(J)$ where $D(J)$ is the free distributive lattice on J and two distinct elements of I are sent to element of disjoint support.

If we take away the disjoint support restriction we add the diagonal operation.

All this has still a geometric interpretation with max and min operations on $[0,1]$.

Finally we can also add an operation corresponding to $1-x$ on $[0,1]$ by replacing distributive lattice by de Morgan algebra.

To add connections allows to reduce the Kan filling operation to the Kan composition operation. This holds provided we have regularity (a lifting of a constant path is constant).

2 Semantics

Types are interpreted by (covariant) presheaves over the following category. The objects are finite sets (of "symbols") and a map $I \to J$ is a set theoretic map $I \to \mathsf{dM}(J)$ where $\mathsf{dM}(J)$ is the *free de Morgan algebra* on J. A de Morgan algebra is a distributive lattice with an operation $1-x$ which satisfies

$$1-0=0 \qquad 1-1=0 \qquad 1-(x\vee y)=(1-x)\wedge(1-y) \qquad 1-(x\wedge y)=(1-x)\vee(1-y)$$

(The difference with Boolean algebras is that we do not require neither $x\wedge(1-x)=0$ nor $x\vee(1-x)=1$.)

There is a "nominal" description of a covariant presheaf over this category. We have a set of elements u which may depends on a finite set of symbols $u(i_1,\dots,i_n)$ and we can do substitutions, replacing $i_1,\dots,i_n$ by elements in $\mathsf{dM}(J)$ for any $J=j_1,\dots,j_m$.

If we restrict ourselves to maps $I \to \mathsf{dM}(J)$ which sends distinct symbols to elements with disjoint support, we get the notion of cubical sets with connections. If we consider arbitrary maps $I \to \mathsf{dM}(J)$ we allow in particular to have the map $i=j$ which corresponds to restriction to a diagonal.

A particular operation that we can do on an element $u=u(i,j,k)$ is to replace i by 0 or 1. We write (ib) this operation for $b=0,1$ so that $u(i0)=u(0,j,k)$ for instance. This corresponds to restriction to a face of u.

3 Remarks on the base category

The base category $\mathcal{C}$ has for objects finite sets of symbols $I,J,K,\dots$ and a morphism $I\to J$ is a map $I\to\mathsf{dM}(J)$. A type will be interpreted as a (covariant) presheaf on $\mathcal{C}$, while a type $A\vdash_I$ depending on symbols in I is interpreted by a presheaf on $I\backslash\mathcal{C}$. That is such a type is a family of sets Af for $f:I\to J$ with restriction maps

$$Af \to Afg \qquad\qquad u \longmapsto ug$$

for $g:J\to K$.

We say that a map $f:I\to J$ is *strict* if if is neither 0 nor 1 for all i in I. One key remark is the following.

Lemma 3.1 *If $f:I\to J$ is strict and ψ in $\mathsf{dM}(I)$ such that $\psi f=b$ (where b is 0 or 1) then $\psi=b$.*

This does not hold for Boolean algebra. For instance the map $(i=j):\{i,j\}\to\{j\}$ is strict and $(i\wedge(1-j))(i=j)=0$ but $i\wedge(1-j)$ is neither 0 nor 1.

Each face map $\alpha:I\to I_\alpha$ is *epi*. If $f:I\to J$ we write $f\leqslant\alpha$ to mean that there exists a map f' (uniquely determined) such that $f=\alpha f'$. This means that $if=i\alpha$ for all i in the domain of α.

Corollary 3.2 *If $fg\leqslant\alpha$ and g is strict then $f\leqslant\alpha$.*

We consider the partial meet semilattice M generated by the face operations (ib). An element of M can be thought as a finite sequence of the form $(i0)(j1)(k0)\dots$. We denote by $\alpha,\beta,\dots$ an element of M. In particular M contains the empty sequence 1_M. There is a canonical partial order on M and a partial product $\alpha\beta$ if α and β are compatible. For instance if $\alpha = (i0)(j1)$ and $\beta = (i0)(k0)$ then $\alpha\beta = (i0)(j1)(k0)$.

Any element α using symbols in I defines a face operation $\alpha : I \to I_\alpha$ where $I_\alpha = I - dom(\alpha)$. Given α, β using symbols in I, we say that α and β are *compatible* if we have $i\alpha = i\beta$ for i in $dom(\alpha) \cap dom(\beta)$. We can then write $\alpha\alpha_1 = \gamma = \beta\beta_1$ where the domain of γ is the union of the domain of α and β. In the poset M the element γ is the meet of α and β.

Any map $f : I \to J$ can be written uniquely as the composition $f = \alpha h$ of a face map $\alpha : I \to I_\alpha$ and a map $h : I_\alpha \to J$ which is strict.

Lemma 3.3 *If we have $\alpha f = \beta g$ with $f : I_\alpha \to J$ and $g : I_\beta \to J$ then α and β are compatible. If γ is the meet of α and β, then there exists a unique $h : I_\gamma \to J$ such that $\alpha f = \gamma h = \beta g$. If we write $\alpha\alpha_1 = \gamma = \beta\beta_1$ then $\alpha_1 f = h = \beta_1 g$.*

Let A be a type depending on symbols in I, so that A is a system of sets Af with restriction maps $Af \to Afg$. Let L be a set of incomparable face operations on I. A *L-system for A* is given by a family a_α in $A\alpha$ which is compatible: if $\alpha\alpha_1 = \beta\beta_1$ then $a_\alpha\alpha_1 = a_\beta\beta_1$. This implies that if $\alpha f = \beta g$ then we have $a_\alpha f = a_\beta g$. We think of such a system as a system of equations $u\alpha = a_\alpha$ for α in L. Notice that any element v in $A1$ defines a compatible system $a_\alpha = v\alpha$.

If L is a set of incomparable face operations on I, we write $\alpha \leqslant L$ to mean that α is $\leqslant$ some element in L. If $f : I \to J$ we define Lf to be the set of maximal face operations β on J such that $f\beta \leqslant L$. In general we have $\beta \leqslant Lf$ iff $f\beta \leqslant L$.

If $f : I \to J$ and we have a L-system a_α we define a corresponding Lf system b_β by taking $b_\beta = a_\alpha g$ whenever $f\beta = \alpha g$. If we have $b_{\beta_0} = a_{\alpha_0} g_0$ and $b_{\beta_1} = a_{\alpha_1} g_1$ it follows from Lemma 9.1 that if β_0 and β_1 are compatible, with $\beta_0\delta_0 = \beta_1\delta_1$ then $b_{\beta_0}\delta_0 = b_{\beta_1}\delta_1$.

4 Composition operation

Since we have added connections we need to generalize the operation of Kan composition.

This can be seen for the following example. If we define the following

$$a_{i1} = \mathsf{comp}^i_{A,\vec{u}}(a_{i0})$$

where $\vec{u}$ is the system u_{j0}, u_{j1} such that $a_{i0}(j0) = u_{j0}(i0)$ and $a_{i0}(j1) = u_{j1}(i0)$, then a_{i1} can be seen as a "line" in the direction j connecting $u_{j0}(i1)$ to $u_{j1}(i1)$. What happens if we do the substitution f which replaces j by $k \wedge l$? For this, we need to introduce a new kind of composition

$$a_{i1}f = \mathsf{comp}^i_{Af,\vec{u'}}(a_{i0}f)$$

where $\vec{u'}$ is the system

$$u'_{k0} = u_{j0} \quad u'_{l0} = u_{j0} \qquad u'_{(k1)(l1)} = u_{j1}$$

Similarly, if g is the substitution that replaces j by $k \vee l$ we get

$$a_{i1}g = \mathsf{comp}^i_{Ag,\vec{u''}}(a_{i0}g)$$

where $\vec{u''}$ is now the system

$$u''_{(k0)(l0)} = u_{j0} \quad u''_{k1} = u_{j1} \qquad u''_{l1} = u_{j1}$$

In this way, we need to consider not only simple face operations $(i0)$ or $(i1)$ but composition of these operations of the form $(i0)(j1)(k0)$.

A *L-system* will be defined to be a finite set L of pairwise incomparable elements α of M and compatible conditions of the form $a\alpha = t_\alpha$.

For instance, if L consists of $(i0), (j0), (i1)(j1)$ then $L(i0)$ is 1_M and $L(i1)$ is $(j0), (j1)$. (All elements in $L(ib)$ are independent of i.) If L is independent of i then $L(ib)$ is the same as L.

If we have a L-system $\vec{u}$ on i, I and we have a map $f : I \to \mathsf{dM}(J)$ with i not in J we can define a system $\vec{u}f$ on i, J. For instance if $\vec{t}$ is a L-system we define $\vec{t}(ib)$ to be the conditions $a\alpha = t_\alpha(ib)$ for α independent of i and $a\gamma = t_{\gamma(ib)}$ if $(ib)\gamma$ is in L. This is a system of $A(ib)$. For another example, if we consider the system a_{i0}, a_{i1} and the substitution $i = i \wedge j$ we obtain the system $b_{i0} = a_{i0},\ b_{j0} = a_{i0},\ b_{(i1),(j1)} = a_{i1}$.

Here is an example (due to Georges Gonthier) which shows the problem if we use a Boolean algebra instead of a de Morgan algebra. Consider the system $u(i0) = a_{i0}$. If we take $i = j \wedge k$ the system becomes $u(j0) = a_{i0},\ u(k0) = a_{i0}$. If we then take $k = 1 - j$ we get $u(j0) = a_{i0},\ u(j1) = a_{i0}$. But if we take directly $i = j \wedge (1 - j) = 0$ then we get the system $u = a_{i0}$ instead. So there is a coherence problem.

An element a in A is *compatible* with or *satisfies* the J-system $\vec{u}$ if we have $a\alpha = u_\alpha$ for all α in J. We can then consider a to be a solution of the constraints defined by $\vec{u}$.

We only have one operation ($0, 1$ plays a symmetric role)

$$\mathsf{comp}^i_{A,\vec{u}}(a_{i0}) : A(i1)$$

where L *is independent* of i and where $a_{i0} : A(i0)$ is an element independent of i and satisfying the L-system $\vec{u}(i0)$ and which produces an element satisfying the L-system $\vec{u}(i1)$.

The symbol i is bound in this operation (but i may occur in A and u_α).

This operation should be *regular* in the sense that

$$\mathsf{comp}^i_{A,\vec{u}}(a_{i0}) = a_{i0}$$

whenever $A, \vec{u}$ is independent of i.

From this operation we can define

$$\tilde{a} = \mathsf{fill}^i_{A,\vec{u}}(a_{i0}) = \mathsf{comp}^j_{A(i\wedge j),\vec{u}(i\wedge j)}(a_{i0})$$

element of type A which satisfies $\tilde{a}(i0) = a_{i0}$ by *regularity* and $\tilde{a}\alpha = u_\alpha(i \wedge j)(j1) = u_\alpha$.

The *uniformity condition* which is required is that we have if $f : I \to \mathsf{dM}(J)$ and I contains the free symbols of $\mathsf{comp}^i_{A,\vec{u}}(a_{i0})$

$$\mathsf{comp}^i_{A,\vec{u}}(a_{i0})f = \mathsf{comp}^j_{Ag,\vec{u}g}(a_{i0}f)$$

where $g : I, i \to \mathsf{dM}(J, j)$ is any extension of f with $g(i) = j$ not in J (which reflects that i is bound in this operation and a_{i0} is independent of i).

Lemma 4.1 *If we have a L-system t_α of A and a_{i0} in $A(i0)$ and both u and v in A satisfy*

$$u\alpha = v\alpha = t_\alpha \qquad u(i0) = v(i0) = a_{i0}$$

then there is a L-path between $u(i1)$ and $v(i1)$.

Proof. We introduce a fresh symbol j and define a $L, (j0), (j1)$-system $\vec{w}$ by taking $w_\alpha = t_\alpha$ and $w_{j0} = u$ and $w_{j1} = v$. We can then consider $\mathsf{comp}^i_{A,\vec{w}}(a_{i0})$ which is a L-path between $u(i1)$ and $v(i1)$. □

Here is a special case of the composition operation which will be convenient.

Lemma 4.2 *Given a type A and a in A and a L-system of "lines" $t_\alpha : a\alpha \to u_\alpha$ in A, there exists a' in A such that $a'\alpha = u_\alpha$ for all α in L.*

Proof. Let i be a fresh symbol. We can define $a' = \mathsf{comp}^i_{A,\vec{v}}(a)$ where $v_\alpha = t_\alpha i$, so that $v_\alpha(i0) = a\alpha$ and $v_\alpha(i1) = u_\alpha$. □

We write $a' = \mathsf{comp}(\vec{t}, a)$. Notice that, by regularity, we have $\mathsf{comp}(\vec{t}, a) = a$ if all lines t_α are constant.

5 Contractible types

A type A is *contractible* iff we can solve in an uniform way any L-system in A.

For instance the free Boolean algebra on the set of symbols is contractible. This follows from the fact that there is exactly one way to fill a cube given its corner. For instance, since we have $a = (1 - i)a(i0) + ia(i1)$ any line $u : u_0 \to_i u_1$ has to be $u = (1 - i)u_0 + iu_1$.

6 Equivalence

We say that $\sigma : T \to A$ is an *equivalence* if, given a L-system $\vec{t}$ in T and a in A compatible with $\sigma\vec{t}$, we can find t in T compatible with $\vec{t}$ with a L-path between σt and a. (This implies that σ has a homotopy inverse; the next Lemma shows that to be an equivalence is in fact logically equivalent to having a homotopy inverse.)

Lemma 6.1 *If σ has a homotopy inverse then σ is an equivalence.*

This generalizes slightly the «graduate lemma», and it has a rather direct proof.

Proof. We assume given $\delta : T \to A$ and $\eta a : \sigma\delta a \to a$ and $\epsilon t : \delta\sigma t \to t$. We assume given t_α and a in A such that $a\alpha = \sigma\alpha t_\alpha$ for all α in L.

We introduce a fresh symbol i and define first by Kan filling θ in T such that

$$\theta = \mathsf{fill}^i_{T,\vec{t}}(\delta a)$$

where $t_\alpha = \epsilon\alpha t_\alpha i$, so that θ satisfies

$$\theta(i0) = \delta a \qquad \theta\alpha = \epsilon\alpha t_\alpha i$$

and the composition $t = \theta(i1)$ is such that $t\alpha = t_\alpha$ for all α in L.

We introduce next a fresh symbol j and we define the system $\vec{v}$ over $L, (i0), (i1)$ by taking

$$v_{i1} = \epsilon t j \qquad v_{i0} = \delta a \qquad v_\alpha = \epsilon t_\alpha (i \wedge j)$$

If we define θ' by (reverse) composition over this system from θ we have

$$\theta'(i0) = \delta a \qquad \theta'(i1) = \delta\sigma t \qquad \theta'\alpha = \delta\alpha(\sigma\alpha t_\alpha)$$

We define the system

$$w_{i0} = \eta a \qquad w_{i1} = \eta\sigma t \qquad w_\alpha = \eta\alpha(\sigma\alpha t_\alpha)$$

and

$$\theta'' = \mathsf{comp}^j_{A,\vec{w}}(\sigma\theta')$$

which is a L-path between σt and a. □

Notice that if σ, δ are the identity functions and ϵ, η the constant path function, then $t = a$ and the L-path between $\sigma t = a$ and a is the constant path a.

For the composition in the universe we need the following fact.

Lemma 6.2 *If $E : T \to_i A$ and we define $\sigma : T \to A$ by $\sigma a = \mathsf{comp}^i_E(a)$ then σ is an equivalence.*

Proof. We assume given t_α and a in A such that $a\alpha = \sigma\alpha t_\alpha$ for all α in L.

We define

$$\theta = \mathsf{fill}^{1-i}_{E,\vec{v}}(a)$$

with $v_\alpha = \mathsf{fill}^i_{E\alpha}(t_\alpha)$. If $t = \theta(i0)$ in T we have $t\alpha = t_\alpha$ for α in L.

We also have $\theta' = \mathsf{fill}^i_E(t)$ such that $\theta'(i1)\alpha = \sigma\alpha t_\alpha$.

We can then consider

$$v = \mathsf{comp}^i_{E,\vec{w}}(t)$$

where $\vec{w}$ is a $L, (j0), (j1)$-system with $w_\alpha = v_\alpha$ and $w_{j0} = \theta'$ and $w_{j1} = \theta$ for a fresh j, so that v is a L-path between σt and a in the direction j. □

7 Representation of cubical sets

If we have a well-founded set X, we define what is a X*-set*. This is inductively given by a family of predicates A_α for α in X where A_α is a set-valued predicate on the set of sequences $a_\beta,\ \beta < \alpha$ such that a_β is in $A_\beta(a_\gamma)_{\gamma<\beta}$. To a given X-set A_α we can associate the set T of sequences (a_α) such that a_α is in $A_\alpha(a_\beta)_{\beta<\alpha}$. If we let T_α be the set of sequences $a_\beta,\ \beta < \alpha$ such that a_β is in $A_\beta(a_\gamma)_{\gamma<\beta}$, then A_α is a family of sets over T_α. We also have a canonical restriction map $T \to T_\alpha$ for all α in X and $T_\alpha \to T_\beta$ if $\beta < \alpha$.

For instance if X is the ordinal $0 < 1 < 2$ then a X-set consists in a set A_0 with a family of sets A_1 over A_0 and a family of sets $A_2(a_0, a_1)$ for a_0 in A_0 and a_1 in $A_1(a_0)$.

It also can be noticed that X-sets form in a natural way a model of type theory with Π, Σ and universes. For instance, for $X = 0 < 1$ we get the model where a type is interpreted by a set A_0 together with a family of sets $A_1(a_0)$.

Let X_I be the finite (and hence well-founded) poset of face operations on I. If for instance $I = i, j$ then a X_I set consists in 4 sets $A_{ib,jc}$, and set-valued relations $A_{ib}(x_{ib,j0}, x_{ib,j1})$ and $A_{jc}(x_{i0,jc}, x_{i1,jc})$ and a relation $A_1(x_{i0,j0}, x_{i0,j1}, x_{i1,j0}, x_{i1,j1}, x_{i0}, x_{i1}, x_{j0}, x_{j1})$. We shall write simply I-sets instead of X_I-sets.

We refine the semantics of a closed type $A \vdash_I$. It is given by a family of sets Af for $f : I \to J$ and restriction maps $Af \to Afg$ if $g : J \to K$. We require Af to be a J-set, and $Af \to Afg$ is the canonical restriction map if g is a face map $\beta : J \to J_\beta$.

It can be checked that all type-forming operations produce objects of this form. For instance if F and G are any presheaves on $\mathcal{C}$ then $G^F(I)$ is a set of sequences λ_f in $F(J) \to G(J)$ for $f : I \to J$, satisfying the condition $(\lambda_f\ u)g = \lambda_{fg}\ ug$, and this has a natural structure of I-set.

We define the following operation on I-sets. Let A be a I-set and T_α be a L-system of I_α-sets, with compatible maps $\sigma_\alpha : T_\alpha \to A\alpha$. Then we can consider the I-set $(\vec{T}, A)$, the element of which are sequences (u_α) where u_α is in T_α if $\alpha \leqslant L$ and in A_α otherwise. For instance, if $I = i$ and A is given by A_{i0}, A_{i1}, A_1 and we have $\sigma_{i0} : T_{i0} \to A_{i0}$ then $(\vec{T}, A)$ is the set of sequences (t_{i0}, a_{i1}, a_1) such that a_1 is in $A_1(\sigma_{i0} t_{i0}, a_{i1})$. It is natural to write the elements on the form $(\vec{t}, a)$.

The key remark is then that if each σ_α is the identity map we have $(\vec{T}, A) = A$ and we have $(\vec{t}, a) = a$ if we take $t_\alpha = a\alpha$.

This basic operation will be used to define *glueing* (which transforms equivalence to equality) and the *composition* operation in the universe. In each case, we will get the same underlying type if all maps are identities. In the case of glueing however, the Kan composition operations does not need to stay the same, while it will be the same for composition, which ensures regularity for composition in the universe.

If we have $B = (\vec{T}, A)$ then there is a canonical map of I-sets $\delta : B \to A$. For instance, if $I = i$ and A is given by A_{i0}, A_{i1}, A_1 and we have $\sigma_{i0} : T_{i0} \to A_{i0}$ then B is the set of sequences (t_{i0}, a_{i1}, a_1) such that a_1 is in $A_1(\sigma_{i0} t_{i0}, a_{i1})$ and we define $\delta(t_{i0}, a_{i1}, a_1) = (\sigma_{i0} t_{i0}, a_{i1}, a_1)$.

8 Glueing operation

In order to interpret univalence we explain how to transform an equivalence to an equality.

Given a L-system $\vec{T}$ in U and a type A together with a compatible system of equivalences $\sigma_\alpha : T_\alpha \to A\alpha$, we define a new type

$$B = \mathsf{glue}(\vec{T}, A, \vec{\sigma})$$

As a cubical set B is $(\vec{T}, A)$. An element of this type is of the form $(\vec{t}, a)$ with $t_\alpha : T_\alpha$ and $a : A$ such that $a\alpha = \sigma_\alpha t_\alpha$. If $L = \{1_M\}$, we have only one equivalence $\sigma_1 : T_1 \to A$ and we take $B = T_1$. If L is empty we take $B = A$. If $f : I \to J$ we define

$$Bf = \mathsf{glue}(\vec{T}f, Af, \vec{\sigma}f)$$

If we have one equivalence $\sigma : T \to A$, then introducing a fresh symbol i, we have $A = A(i0)$ and $B = \mathsf{glue}(T_{i0}, A, \sigma_{i0})$ with $T_{i0} = T$ and $\sigma_{i0} = \sigma$. This type B will be such that $B(i0) = T$ and $B(i1) = A(i1) = A$. So we have an operation transforming an equivalence to an equality. (We are only interested in this case, but since we have to define B as a presheaf we need to consider the general case as well.)

We have a map $\delta : B \to A$ defined by $\delta(\vec{t}, a) = a$ and $\delta\alpha t = \sigma_\alpha t$ for α in L.

The main algorithm is to define $\mathsf{comp}^i_{B,\vec{v}}(v_{i0})$ for a J-system $\vec{v}$. Using the map δ, we define a J-system $\vec{u}$ in A.

We consider v_{i0} and $a_{i0} = \delta(i0)v_{i0}$ in $A(i0)$ and

$$a_{i1} = \mathsf{comp}^i_{A,\vec{u}}(a_{i0}) : A(i1)$$

For β in J we have $a_{i1}\beta = u_\beta(i1)$.

The goal is to build

$$v_{i1} = \mathsf{comp}^i_{B,\vec{v}}(v_{i0}) : B(i1)$$

We let L' be the subset of γ in L not mentionning i and L'' subset of γ such that $\gamma(i1)$ is in L. Since the element in L are incomparable, L' and L'' are disjoint but it may be that some element in L' is $<$ some element in L''. We have that $L(i1)$ is a the union of L'' and of L'_1 subset of element in L' not $< L''$.

We want to write $v_{i1} = (\vec{r}, u_{i1})$ for some u_{i1} in $A(i1)$ which should be obtained from a_{i1} by modifying some faces in L' and L''. What are the constraints on this element u_{i1}?

For γ in L', we should have $u_{i1}\gamma = \sigma_\gamma(i1)t_{i1,\gamma}$ where $t_{i1,\gamma} = \mathsf{comp}^i_{T_\gamma,\vec{v}\gamma}(v_{i0}\gamma)$ is of type $T_\gamma(i1)$.

For γ in L'', then $u_{i1}\gamma$ should be of the form $\sigma_{\gamma(i1)}r_\gamma$ for some r_γ in $T_{\gamma(i1)}$.

We first deal with the constraints for γ in L'. We have

$$a_{i0}\gamma = \sigma_\gamma(i0)v_{i0}\gamma \qquad a_{i1}\gamma = \mathsf{comp}^i_{A\gamma,\vec{u}\gamma}(a_{i0}\gamma) \qquad t_{i1,\gamma} = \mathsf{comp}^i_{T_\gamma,\vec{v}\gamma}(v_{i0}\gamma)$$

We can build a line $w_\gamma : \sigma_\gamma(i1)t_{i1,\gamma} \to a_{i1}\gamma$ using Lemma 4.1 since both are two compositions in the type $A\gamma$ of the same shape, namely a composition from $a_{i0}\gamma$ using $\vec{u}\gamma$.

(There is no reason for this line to be constant even if σ_γ is the identity map. This is why we need another argument for composition in the universe.)

We apply Lemma 4.2, defining $a'_{i1} = \mathsf{comp}(\vec{w}, a_{i1})$.

Notice that all the lines $w_\gamma\delta$ are constant for δ in $J\gamma$. By regularity, we have $a'_{i1}\beta = a_{i1}\beta$ for β in J.

The second step deals with the element γ in L''. For such an element γ we have $\alpha = \gamma(i1)$ in L.

For β in $J\gamma$ we have $\gamma\beta \leqslant J$ and so $a'_{i1}\gamma\beta = \sigma_\alpha\beta t_\beta$ for some t_β in $T_\alpha\beta$. Similarly, for β in $L'\gamma$ we have $\gamma\beta \leqslant L'$ and we also have $a'_{i1}\gamma\beta = \sigma_\alpha\beta t_\beta$ for some t_β in $T_\alpha\beta$. Since σ_α is an equivalence we can build t_γ in $T_{\gamma(i1)} = T_\alpha$ and a line $s_\gamma : \sigma_\alpha t_\gamma \to a'_{i1}\gamma$ which is a $(L' \cup J)\gamma$-path. We change then a'_{i1} to $u_{i1} = \mathsf{comp}(\vec{s}, a'_{i1})$ again using Lemma 4.2.

By regularity, we have $u_{i1}\delta = a'_{i1}\delta$ for δ in $L' \cup J$, so we did not modify the $L' \cup J$ faces of a'_{i1}.

The element $u_{i1} = \mathsf{comp}(\vec{s}, a'_{i1})$ satisfies $u_{i1}\gamma = \sigma_\gamma t_{i1,\gamma}$ for γ in L' and $u_{i1}\gamma = \sigma_{\gamma(i1)}t_\gamma$ for γ in L''. Hence, we can define a corresponding element $v_{i1} = (\vec{r}, u_{i1})$ in $B(i1)$ with $r_\gamma = t_{i1,\gamma}$ in $T_\gamma(i1)$ for γ in L'_1 and $r_\gamma = t_\gamma$ in $T_{\gamma(i1)}$ for γ in L''.

9 Composition in the universe

It is almost the same operation as for glueing. We assume given a type A independent of j and types E_α such that $E_\alpha(j0) = A\alpha$ and we explain how to build

$$B = \mathsf{comp}^j_{U,\vec{E}}(A)$$

We see the type E_α as an equality between $A\alpha = E_\alpha(j0)$ and $T_\alpha = E_\alpha(j1)$. We have a corresponding equivalence $\sigma_\alpha : T_\alpha \to A\alpha$ using Lemma 6.1. The two types B and $\mathsf{glue}(\vec{T}, A, \vec{\sigma})$ are the same as cubical sets but don't have the same Kan composition operations.

Lemma 9.1 *We assume given A, T, E with $E(j0) = T$ and $E(j1) = A$ and we define $\sigma : T \to A$ by $\sigma t = \mathsf{comp}^j_E(t)$. Given a L-system $\vec{t}$ in T and t_{i0} in $T(i0)$ solution of $\vec{t}(i0)$ we can define the L-system $a_\alpha = \sigma\alpha t_\alpha$ in A and $a_{i0} = \sigma(i0)t_{i0}$. We define $a_{i1} = \mathsf{comp}^i_{A,\vec{a}}(a_{i0})$ and $t_{i1} = \mathsf{comp}^i_{T,\vec{t}}(t_{i0})$. We can build a $L(i1)$-path $\sigma(i1)t_{i1} \to a_{i1}$ which is constant if E is independent of j.*

Proof. We define a L-system in E $e_\alpha = \mathsf{fill}^j_{E\alpha}(t_\alpha)$ and $e_{i0} = \mathsf{fill}^j_{E(i0)}(t_{i0})$ so that

$$e_\alpha(j0) = t_\alpha \quad e_\alpha(j1) = a_\alpha \quad e_{i0}(j0) = t_{i0} \quad e_{i0}(j1) = a_{i0}$$

If $e_{i1} = \mathsf{comp}^i_{E,\vec{e}}(e_{i0})$ we also have $e_{i1}(j0) = t_{i1}$ and $e_{i1}(j1) = a_{i1}$.

We define $e'_{i1} = \mathsf{fill}^j_E(t_{i1})$ so that $e'_{i1}(j0) = t_{i1}$ and $e'_{i1}(j1) = \sigma(i1)t_{i1}$. If k is a fresh symbol, we take $u_{k0} = e'_{i1}$ and $u_{k1} = e_{i1}$ and

$$\tilde{a_{i1}} = \mathsf{comp}^j_{E(i1),\vec{u}}(t_{i1})$$

We have then $\tilde{a_{i1}}(k0) = \sigma(i1)t_{i1}$ and $\tilde{a_{i1}}(k1) = a_{i1}$. Furthermore if E is independent of j then $e'_{i1} = t_{i1}$ by regularity. We have $e_\alpha = t_\alpha$ $e_{i0} = t_{i0}$ as well by regularity, so that $e_{i1} = t_{i1}$. It follows that $\tilde{a_{i1}} = t_{i1}$ also by regularity. □

As for glueing, we define an element of B to be of the form $(\vec{t}, a)$ with a in A and t_α in T_α such that $\sigma_\alpha t_\alpha = a\alpha$.

The operation

$$\mathsf{comp}^i_{B,\vec{v}}(v_{i0})$$

is almost the same as for glueing. We first build a_α and a_{i0} and $a_{i1} = \mathsf{comp}^i_{A,\vec{a}}(a_{i0})$.

The difference is in the first step. For γ in L', we can consider in the type $T_\gamma(i1)$

$$t_{i1} = \mathsf{comp}^i_{T_\gamma,\vec{v}\gamma}(v_{i0}\gamma)$$

We build a line $\sigma_\gamma(i1)t_{i1} \to a_{i1}\gamma$ not by using Lemma 4.1 but by building directly a line between t_{i1} and $a_{i1}\gamma$ by using Lemma 9.1. This line reduces then to a constant if E_γ is independent of j.

10 Identity type

The Kan operation for identity type is similar to the one in [1].

11 Function extensionality

Given f, g of type $\Pi\ A\ F$ and p such that $p\ u : f\ u \to g\ u$ we compute $\mathsf{ext}(i, p) : f \to_i g$ for a fresh i. This amounts to define $\mathsf{ext}(\varphi, p) : \Pi\ A\ F$ for a de Morgan formula φ. Since we don't have linearity condition, we define $\mathsf{ext}(\varphi, p)u = p\ u\ \varphi$.

12 Propositional truncation

13 Operational semantics

We limit ourselves to the description of the system without universes. The point is to explain how we can justify *function extensionality* without using function extensionality at the metalevel.

The syntax for the terms is

$$t, A, F\ ::=\ x \mid t\ t \mid \lambda x.t \mid \mathsf{Id}\ A\ t\ t \mid \Pi\ A\ F \mid \langle i\rangle t \mid \mathsf{comp}^i_{A,\vec{t}}(t) \mid \mathsf{ext}\ t\ t\ t \mid \varphi$$

where φ represents an element in the free de Morgan algebra on the symbols. In this syntax, $\langle i\rangle t$ represents the path abstraction operation, and binds the symbol i. Similarly, $\mathsf{comp}^i_{A,\vec{t}}(t)$ represents Kan composition; it binds the symbol i and $\vec{t}$ represents a system of terms. For instance $\vec{t}$ may be of the form t_{j0}, t_{j1} or of the form $t_{j0}, t_{k0}, t_{(j1)(k1)}$. It may also be empty, in which case we write simply $\mathsf{comp}^i_A(t)$.

We have the usual β-reduction rule

$$(\lambda x.t)\ u = t(x = u)$$

We write $(x : A) \to B$ for $\Pi\ A\ (\lambda x.B)$. If p represents a proof of $(x : A) \to \mathsf{Id}\ B\ (t\ x)\ (u\ x)$ then $\mathsf{ext}\ t\ u\ p$ should be a proof a $\mathsf{Id}\ ((x : A) \to B)\ f\ g$. Its computation rule is then

$$\mathsf{ext}\ t\ u\ p\ \varphi\ v = p\ v\ \varphi$$

If $f : I \to \mathsf{dM}(J)$ we can define the operation $t \longmapsto tf$ on terms. (Notice that this is an operation on terms; we don't have a term constructor for substitution.) It behaves like ordinary substitution, and we have for instance

$$(\langle i \rangle t)f = \langle j \rangle tg$$

where $g : I, i \to \mathsf{dM}(J, j)$ extends f by $g(i) = j$ not in J. We also have

$$(\lambda x.t)f = \lambda x.tf \qquad xf = x \qquad (t\ u)f = tf\ uf \qquad (\Pi\ A\ F)f = \Pi\ Af\ Ff$$

We have already defined $\mathsf{comp}^i_{A,\vec{t}}(t)f$. For instance we have

$$\mathsf{comp}^i_{A,t_{j0},t_{j1}}(a_{i0})f = \mathsf{comp}^i_{Af,u_{k0},u_{l0},u_{(k1)(l1)}}(a_{i0}f)$$

if $jf = k \wedge l$ and $u_{k0} = u_{l0} = t_{j0}$ and $u_{(k1)(l1)} = t_{j1}$. We also have

$$\mathsf{comp}^i_{A,t_{j0},t_{j1}}(a_{i0})(j0) = t_{j0}(i1) \qquad \mathsf{comp}^i_{A,t_{j0},t_{j1}}(a_{i0})(j1) = t_{j1}(i1)$$

We can then state the path reduction law

$$(\langle i \rangle t)\ \varphi = t(i = \varphi)$$

A canonical object of type $\mathsf{Id}\ A\ a\ b$ is of the form $\langle i \rangle t$ with $t(i0) = a$ and $t(i1) = b$. If w is of type $\mathsf{Id}\ A\ a\ b$ and j is a fresh symbol, then $w\ j$ is of type A and $w\ 0 = a$ and $w\ 1 = b$.

The main new computation rules are for the composition of a product type and the composition of an identity type.

For product types, we have

$$\mathsf{comp}^i_{\Pi\ A\ F,\vec{\mu}}(\lambda_{i0})\ u_{i1} = \mathsf{comp}^i_{F\ u,\vec{v}}(\lambda_{i0}\ u(i0))$$

with $u = \mathsf{fill}^{1-i}_A(u_{i1})$ and $v_\alpha = \mu_\alpha\ u\alpha$.

If we add sum types

$$t, A, F\ ::=\ \Sigma\ A\ F \mid (t, t) \mid t.1 \mid t.2$$

we have

$$\mathsf{comp}^i_{\Sigma\ A\ F,\vec{w}}(a_{i0}, b_{i0}) = (a_{i1}, \mathsf{comp}^i_{F\ u,\vec{v}}(b_{i0}))$$

with $a = \mathsf{fill}^i_{A,\vec{u}}(a_{i0})$ and $a_{i1} = a(i1)$ and $u_\alpha = w_\alpha.1$ and $v_\alpha = w_\alpha.2$.

For identity types, we have

$$\mathsf{comp}^i_{\mathsf{Id}\ A\ a\ b,\vec{w}}(w_{i0}) = \langle j \rangle \mathsf{comp}^i_{A,\vec{u}}(w_{i0}\ j)$$

where j is a fresh symbol and $\vec{u}$ is the system defined by $u_\alpha = w_\alpha\ j$ and $u_{j0} = a$ and $u_{j1} = b$.

Using the composition operation we can interpret

$$\mathsf{Id}\ A\ a_0\ a_1 \to B(a_0) \to B(a_1)$$

Indeed if p is of type $\mathsf{Id}\ A\ a_0\ a_1$ and $b_0 : B(a_0)$ then $\mathsf{comp}^i_{B(p\ i)}(b_0)$ is of type $B(a_1)$.

Using the operation $i \wedge j$ on symbols we can interpret the fact that

$$(\Sigma x : A)\mathsf{Id}\ A\ a\ x$$

is contractible. Indeed, if x, p is an element of this type then $q = \langle i \rangle (p\ i, \langle j \rangle p\ (i \wedge j))$ is a path such that $q\ 0 = (a, \langle j \rangle a)$ and $q\ 1 = (x, \langle j \rangle p\ j) = (x, p)$. If $x = a$ and $p = \langle i \rangle a$ (which interprets reflexivity) we get $q = \langle i \rangle (a, \langle j \rangle a)$ which is a constant path.

We can then interpret the usual J elimination rule. Because of the regularity condition, the computation rule for J is interpreted as a judgemental equality.

We can add

$$t, A, F \ ::= \ \mathsf{N} \mid \mathsf{zero} \mid \mathrm{s}(t) \mid \mathsf{natrec}\ F\ t\ t$$

with the usual computation rules

$$\mathsf{natrec}\ F\ a\ f\ \mathsf{zero} = a \qquad \mathsf{natrec}\ F\ a\ f\ \mathrm{s}(n) = f\ n\ (\mathsf{natrec}\ F\ a\ f\ n)$$

The computation rules for $\mathsf{comp}^i_{N,\vec{u}}(u_{i0})$ are the following. First we have

$$\mathsf{comp}^i_{N,\vec{u}}(u_{i0}) = \mathsf{zero}$$

if $u_{i0} = \mathsf{zero}$ and $\vec{u}$ is the constant system $u_\alpha = \mathsf{zero}$. Second we have

$$\mathsf{comp}^i_{N,\vec{u}}(u_{i0}) = \mathrm{s}(\mathsf{comp}^i_{N,\vec{v}}(v_{i0}))$$

if $u_{i0} = \mathrm{s}(v_{i0})$ and $u_\alpha = \mathrm{s}(v_\alpha)$.

14 Typing rules

We have judgement of the forms $\Gamma \vdash_I$, $\Gamma \vdash_I \vdash A$ and $\Gamma \vdash_I t : A$ relativized at a "level" (finite set of symbols) I . The rules are the usual rules of type theory at all levels I, with the *restriction rule*

$$\frac{\Gamma \vdash_I t : A}{\Gamma f \vdash_J tf : Af}$$

if $f : I \to \mathsf{dM}(J)$.

The new rules are then the following.

$$\frac{\Gamma \vdash_I \qquad \Gamma \vdash_{I,i} A \qquad \Gamma \vdash_I a_{i0} : A(i0) \qquad \Gamma\alpha \vdash_{I\alpha,i} u_\alpha : A\alpha}{\Gamma \vdash \mathsf{comp}^i_{A\vec{u}}(a_{i0}) : A(i1)}$$

with $I_\alpha = I - dom(\alpha)$. We also have

$$\frac{\Gamma \vdash_I A \qquad \Gamma \vdash_I a_0 : A \qquad \Gamma \vdash_I a_1 : A}{\Gamma \vdash_I \mathsf{Id}\ A\ a_0\ a_1}$$

$$\frac{\Gamma \vdash_I A \quad \Gamma \vdash_I a_0 : A \quad \Gamma \vdash_I a_1 : A \quad \Gamma \vdash_{I,i} t : A \quad \Gamma \vdash_I t(i0) = a_0 : A \quad \Gamma \vdash_I t(i1) = a_1 : A}{\Gamma \vdash_I \langle i \rangle t : \mathsf{Id}\ A\ a_0\ a_1}$$

In particular, we get the reflexivity proof of $a : A$ by defining $refl\ a$ as the constant path function $\langle i \rangle a$

$$\frac{\Gamma \vdash_I t : \mathsf{Id}\ A\ a_0\ a_1}{\Gamma \vdash_{I,i} t\ i : A}$$

$$\frac{\Gamma \vdash_I t : (x : A) \to B \qquad \Gamma \vdash_I u : (x : A) \to B \qquad \Gamma \vdash_I p : (x : A) \to \mathsf{Id}\ B\ (t\ x)\ (u\ x)}{\Gamma \vdash_I \mathsf{ext}\ t\ u\ p : \mathsf{Id}\ ((x : A) \to B)\ t\ u}$$

15 General remarks about the model

The first remark is that all paths in N are constant, as expected.

Proposition 15.1 $\mathbf{I}$ *is the presheaf defined by* $\mathbf{I}(J) = \mathsf{dM}(J)$ *and* N *is the constant presheaf* $\mathsf{N}(J) = \mathsf{N}$. *Any natural transformation* $\mathbf{I} \to \mathsf{N}$ *is constant and is determined by the image of* i *by the map* $\mathbf{I}(\{i\}) \to \mathsf{N}$.

The second remark is that one cannot hope to have the right lifting property for monomorphisms against trivial fibrations. Indeed, if we had this property, we could do the following operation. For any map $f : A \to B$ if we have $a : A$ and $b : B$ and a path $f\ a \to b$ then we can find $g : A \to B$ such that $g\ a = b$ with a path $f \to g$. (I learnt this from Vladimir Voevodsky.) Indeed it is enough to consider the trivial fibration $(\Sigma y : B)\mathsf{Id}_B\ (f\ x)\ y,\ x : A$ and the monomorphism $1 \to A$ defined by $a : A$. However, it is not possible to have such a map g in general as is shown by the following Kripke model over $0 \leqslant 1$. At time 0 let A have two distinct points a and a' which becomes equal at time 1. Let B be the groupoid having two connected component $u \to b$ and u' at time 0 and only one $u' = u \to b$ at time 1. We then have a map $f : A \to B$ taking $f\ a = u$ and $f\ a' = u'$ and we have a path $f\ a \to b$, but there is no map $g : A \to B$ such that $g\ a = b$ with a path $f \to g$.

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