Variation on Cubical sets

September 9, 2014

Introduction

In the model presented in [1, 4] a type is interpreted by a nominal set A equipped with two "face" operations: if $u : A$ and i is a symbol we can form $u(i = 0) : A(i = 0)$ and $u(i = 1) : A(i = 1)$ elements independent of i. The unit interval is represented by the nominal set $\mathbf{I}$, whose elements are $0, 1$ and the symbols. The set $\mathbf{I}$ however does not satisfy the Kan filling condition.

In this model, if A represents a type, the path space of A is represented by the affine exponential $\mathbf{I} \to *A$, which is adjoint to the separated product $B * I$ with $(b, i) \in B * I$ if i is independent of b.

We modify this model by adding the operations $i \wedge j$ and $i \vee j$ in the set $\mathbf{I}$. This corresponds to adding *connections* [2].

In this way we can reduce the Kan filling operation to the simpler composition operations and we can interpret more definitional equalities. In particular the computation of the elimination rule for equality is interpreted in a definitional way.

1 Connections

In this note we will adopt the following notations. We will use the letters $i, j, k, \ldots$ for symbols/colours/names of $\mathbf{I}$ $b, c \ldots$ for $0, 1$. We write simply ib for the substitution $i = b$ so that $u(ib) : A(ib)$ if $u : A$. We also have a special operation $(i \wedge j)$ corresponding to the connection $i = i \wedge j$. This operation satisfies

$$u(i \wedge j)(i0) = a(i0) = u(i \wedge j)(j0) \qquad u(i \wedge j)(i1) = u(ij) \qquad u(i \wedge j)(j1) = u$$

Also j is a new free symbol of $u(i \wedge j)$. Dually we have an operation $(i \vee j)$.

More generally we can consider the operations $(i = j \wedge k)$ and $(i = j \vee k)$.

2 Kan composition

We consider the partial meet semilattice M generated by the face operations (ib). An element of M can be thought as a finite sequence of the form $(i0)(j1)(k0)\ldots$. We denote by $\alpha, \beta, \ldots$ an element of M. In particular M contains the empty sequence. If α is in M and $a : A$ we can consider $a\alpha : A\alpha$. There is a canonical partial order on M and a partial product $\alpha\beta$ if α and β are compatible. For instance if $\alpha = (i0)(j1)$ and $\beta = (i0)(k0)$ then $\alpha\beta = (i0)(j1)(k0)$.

A L-*system* will be defined to be a finite set L of pairwise incomparable elements α of M and compatible conditions of the form $a\alpha = t_\alpha$. If $\vec{t}$ is a L-system we define $\vec{t}(ib)$ to be the conditions $a(ib)\alpha = t_\alpha(ib)$ for α compatible with (ib) and $a\alpha = t_\alpha$ if $(ib) \leqslant \alpha$. This is a tube of $A(ib)$. We also can define a tube of $A(i \wedge j)$. The conditions will be $a\alpha = t_\alpha(i \wedge j)$ if α is independent of i and $a\alpha(i0) = t_{\alpha(i0)} = a\alpha(j0)$ and $a\alpha(i1)(j1) = t_{\alpha(i1)}$.

An element a in A is *compatible* with the J-system $\vec{u}$ if we have $a\alpha = u_\alpha$ for all j in J. We can then consider a to be a solution of the constraints defined by $\vec{u}$.

We only have one operation (up to symmetry between $0, 1$)

$$\mathsf{comp}^i_{A,\vec{u}}(a_{i0}) : A(i1)$$

where L is independent of i and where $a_{i0} : A(i0)$ is an element compatible with the L-system $\vec{u}(i0)$ and which produces an element compatible with the L-system $\vec{u}(i1)$.

The symbol i is bound in this operation.

This operation should be *regular* in the sense that

$$\mathsf{comp}^i_{A,\vec{u}}(a_{i0}) = a_{i0}$$

whenever $A, \vec{u}$ is independent of i.

From this operation we can define

$$\tilde{a} = \mathsf{fill}^i_{A,\vec{u}}(a_{i0}) = \mathsf{comp}^j_{A(i\wedge j),\vec{u}(i\wedge j)}(a_{i0})$$

element of type A which satisfies $\tilde{a}(i0) = a_{i0}$ and $\tilde{a}\alpha = u_\alpha(i \wedge j)(j1) = u_\alpha$.

We require further conditions on the composition operations. We should have

$$\mathsf{comp}^i_{A,\vec{u}}(a_{i0})(jb) = \mathsf{comp}^i_{A(jb),\vec{u}(jb)}(a_{i0}(jb))$$

and

$$\mathsf{comp}^i_{A,\vec{u}}(a_{i0})(j \wedge k) = \mathsf{comp}^i_{A(j\wedge k),\vec{u}(j\wedge k)}(a_{i0}(j \wedge k))$$

The first condition implies

$$\mathsf{comp}^i_{A,\vec{u}}(a_{i0})\beta = \mathsf{comp}^i_{A\beta,\vec{u}\beta}(a_{i0}\beta)$$

In the case where β is $\leqslant \alpha$ for some α in L (necessarily unique since the elements of L are incomparable) then this reduces to

$$\mathsf{comp}^i_{A,\vec{u}}(a_{i0})\beta = v_\alpha\beta(i1)$$

If we write $\beta = \alpha\gamma$ (since β as a sequence extends α notice that we have $v_\alpha\beta = v_\alpha\gamma$.

We write $u : a_0 \to_i a_1$ to express $v(i0) = a_0$ and $v(i1) = a_1$. We write $v : a_0 \to a_1$ to express that v can be written $\langle i\rangle u$ with $u : a_0 \to_i a_1$.

If we have $u : A\alpha$ the fact that u is independent of all symbols in α is equivalent to $u = u\alpha$. To say that $t_\alpha,\ \alpha \in L$ is a L-system can be written $t_\alpha\beta = t_\beta\alpha$.

Given two elements a and u that defines the same L-system, i.e. $a\alpha = u\alpha$ for all α, we say that p is a L-path bewteen a and u iff we have $p(i0) = a,\ p(i1) = u$ and $p\alpha = a\alpha = u\alpha$.

Lemma 2.1 *If we have a L-system t_α of A and a_{i0} in $A(i0)$ and both u and v in A satisfy*

$$u\alpha = v\alpha = t_\alpha \qquad u(i0) = v(i0) = a_{i0}$$

then there is a L-path between $u(i1)$ and $v(i1)$.

Proof. We introduce a fresh symbol j and define a $L, (j0), (j1)$-system $\vec{w}$ by taking $w_\alpha = t_\alpha$ and $w_{j0} = u$ and $w_{j1} = v$. We can then consider $\mathsf{comp}^i_{A,\vec{w}}(a_{i0})$ which is a L-path between $u(i1)$ and $v(i1)$. □

3 Equivalence

We say that $\sigma : T \to A$ is an *equivalence* if, given a L-system $\vec{t}$ in T and a in A compatible with $\sigma\vec{t}$, we can find t in T compatible with $\vec{t}$ with a L-path between σt and a.

Lemma 3.1 *If σ has a homotopy inverse then σ is an equivalence.*

This corresponds to the «graduate lemma», but we have a rather direct proof.

Proof. We assume given $\delta : T \to A$ and $\eta a : \sigma\delta a \to a$ and $\epsilon t : \delta\sigma t \to t$. We assume given t_α and a in A such that $a\alpha = \sigma\alpha t_\alpha$ for all α in L.

We introduce a fresh symbol i and define first by Kan filling θ in T such that

$$\theta = \mathsf{fill}^i_{T,\vec{t}}(\delta a)$$

where $t_\alpha = \epsilon\alpha t_\alpha i$, so that θ satisfies

$$\theta(i0) = \delta a \qquad \theta\alpha = \epsilon\alpha t_\alpha i$$

and the composition $t = \theta(i1)$ is such that $t\alpha = t_\alpha$ for all α in L.

We introduce next a fresh symbol j and we define the system $\vec{v}$ over $L, (i0), (i1)$ by taking

$$v_{i1} = \epsilon t j \qquad v_{i0} = \delta a \qquad v_\alpha = \epsilon t_\alpha (i \wedge j)$$

If we define θ' by (reverse) composition over this system from θ we have

$$\theta'(i0) = \delta a \qquad \theta'(i1) = \delta\sigma t \qquad \theta'\alpha = \delta\alpha\sigma t_\alpha$$

We define the system

$$w_{i0} = \eta a \qquad w_{i1} = \eta t \qquad w_\alpha = \eta\alpha t_\alpha$$

and

$$\theta'' = \mathsf{comp}^j_{A,\vec{w}}(\sigma\theta')$$

which is a L-path between a and σt. $\square$

4 Glueing operation

In order to interpret univalence we explain how to transform an equivalence to an equality.

Given a L-system $\vec{T}$ in U and a type A together with equivalences $\sigma_\alpha : T_\alpha \to A\alpha$ we define a new type

$$B = \mathsf{glue}(\vec{T}, A, \vec{\sigma})$$

An element of this type is of the form $(\vec{t}, a)$ with $t_\alpha : T_\alpha$ and $a : A$ such that $a(\alpha) = \sigma_\alpha t_\alpha$. If L consists only of one condition, we take $B = T$. We should have

$$B(ib) = \mathsf{glue}(\vec{T}(ib), A(ib), \vec{\sigma}(ib))$$

and

$$B(i \wedge j) = \mathsf{glue}(\vec{T}(i \wedge j), A(i \wedge j), \vec{\sigma}(i \wedge j))$$

where $\vec{\sigma}(i \wedge j)$ is defined similarly to the corresponding operation on tubes.

In the special case where we have only one equivalence $T \to A$, then we have for i fresh $A = A(i0)$ and $B = \mathsf{glue}(T, A, \sigma)$ will be such that $B(i0) = T$ and $B(i1) = A(i1) = A$. So we have an operation transforming an equivalence to an equality.

We have a map $f : B \to A$ defined by $f(\vec{t}, a) = a$ and $f\alpha t = \sigma_\alpha t$ for α in L.

The main algorithm is to define $\mathsf{comp}^i_{B,\vec{v}}(v_{i0})$ for a J-system $\vec{v}$. We have $v_\beta = (\vec{t}\beta, u_\beta)$. Using the map f, we define a J-system $\vec{u}$ in A.

We consider v_{i0} and $a_{i0} = f(i0)v_{i0}$ in $A(i0)$ and

$$a_{i1} = \mathsf{comp}^i_{A,\vec{u}}(a_{i0}) : A(i1)$$

We may have to change some $a_{i1}\gamma$ for γ in $L(i1)$.

The goal is to build

$$v_{i1} = \mathsf{comp}^i_{B,\vec{v}}(v_{i0}) : B(i1)$$

Let L' be the set of γ in L not mentionning i.

For γ in L' we can consider in the type $T_\gamma(i1)$

$$t_{i1} = \mathsf{comp}^i_{T_\gamma,\vec{v}\gamma}(v_{i0}\gamma)$$

We have $a_{i0}\gamma = \sigma_\gamma(i0)v_{i0}\gamma$. We can build a line $\sigma_\gamma(i1)t_{i1} \to a_{i1}\gamma$ using Lemma 2.1 since both are two compositions of the same shape.

In this case, there is no reason for this line to be constant if σ_γ is the identity map. (We need another argument for composition in the universe.)

We do all these changes by a composition first. We have modified $a_{i1}\gamma$ for γ in L', and should not change it.

The second step deals with $\alpha = (i1)\gamma$ in L. In this case we don't have the element t_{i1} already in T_α. We conclude by considering $J\alpha$ and $L'\alpha$ and noticing that $a_{i1}\alpha\beta$ are in the image of $\sigma_\alpha\beta$ for β in $J\alpha$ and $L'\alpha$ and by the fact that σ_α is an equivalence.

5 Composition in the universe

It is almost the same operation as for glueing. We assume given a type A independent of j and types E_α such that $E_\alpha(j0) = A\alpha$ and we explain how to build

$$B = \mathsf{comp}^j_{U,\vec{E}}(A)$$

We see the type E_α as an equality between $A\alpha = E_\alpha(j0)$ and $T_\alpha = E_\alpha(j1)$. We have a corresponding equivalence $\sigma_\alpha : T_\alpha \to A\alpha$.

We define an element of B to be of the form $(\vec{t}, a)$ with a in A and t_α in T_α such that $\sigma_\alpha t_\alpha = a\alpha$.

The operation

$$\mathsf{comp}^i_{B,\vec{v}}(v_{i0})$$

is almost the same as in the previous case. We first build a_α and a_{i0} and $a_{i1} = \mathsf{comp}^i_{A,\vec{a}}(a_{i0})$.

The difference is in the first step. For γ in L', we can consider in the type $T_\gamma(i1)$

$$t_{i1} = \mathsf{comp}^i_{T_\gamma,\vec{v}\gamma}(v_{i0}\gamma)$$

We build a line $\sigma_\gamma(i1)t_{i1} \to a_{i1}\gamma$ not by using Lemma 2.1 but by building directly a line between t_{i1} and $a_{i1}\gamma$ by a composition in E_γ. This line reduces then to a constant if E_γ is independent of j.

6 Identity type

The Kan operation for identity type is similar to the one in [1].

7 Function extensionality

Given f, g of type $\Pi\ A\ F$ and p such that $p\ u : f\ u \to g\ u$ we compute $\mathsf{ext}(i,p) : f \to_i g$ for a fresh i. This amounts to define $\mathsf{ext}(\varphi, p) : \Pi\ A\ F$ for a boolean formula φ. Given u, we define the system $v_\alpha = f\alpha(u\alpha)$ for $\varphi\alpha = 0$ and $v_\alpha = p\alpha(u\alpha)j$ for $\varphi\alpha = 1$. We then take

$$\mathsf{ext}(\varphi, p)\ u = \mathsf{comp}^j_{A,\vec{v}}(f\ u)$$

There is an alternative definition where we limit the system to $v_\alpha = p\alpha(u\alpha)j$ for $\varphi\alpha = 1$. It is still connecting f to g by regularity.

8 Propositional truncation

References

[1] M.Bezem, Th. Coquand and S. Huber. A model of type theory in cubical sets. Preprint, 2013.

[2] R. Brown, P. J. Higgins and R. Sivera. *Nonabelian Algebraic Topology: Filtered spaces, crossed complexes, cubical homotopy groupoids.* volume 15 of EMS Monographs in Mathematics , European Mathematical Society, 2011.

[3] A. Grothendieck. Pursuing stacks. Manuscript, 1983.

[4] A. M. Pitts. An Equivalent Presentation of the Bezem-Coquand-Huber Category of Cubical Sets. Manuscript, 17 September 2013.