Simplicial sets model of type theory

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Introduction

One important question about Voevodsky's simplicial model of type theory is if there is a constructive version of this model. We give a version of this model where the only non constructive principle which is used is the decidability of degeneracy.

1 Simplicial sets

The notion of simplicial set makes sense in constructive mathematics as a presheaf from the category Δ of finite nonempty linear orders. We write $[0], [1], \dots$ the objects of the category Δ and $\epsilon^i : [n-1] \to [n]$ the injection that skips the value $i \in [n]$ and $\eta^j : [n+1] \to [n]$ the surjection covering $j \in [n]$ twice. Any surjection $[n] \to [p]$ can be written uniquely $\eta^{j_t} \dots \eta^{j_1}$ with $j_t < \dots < j_1 < n$ and $n - m = t$. Also, any map $\alpha : [p] \to [n]$ can be decomposed uniquely as $\alpha = \epsilon\eta$ with $\epsilon : [q] \to [n]$ injective and $\eta : [p] \to [q]$ surjective.

Let X be a simplicial set. Eilenberg-Zilber Lemma states that any $x \in X[n]$ can be written uniquely on the form $x = y\eta$ with $\eta : [n] \to [m]$ surjection and $y \in X[m]$ non degenerate, In a constructive setting, this does not hold since it may not be decidable in general if a given $x \in X[n]$ is degenerate or not.

Definition 1.1 We say that a simplicial set X is *decidable* iff it is decidable whether or not $x \in X[n+1]$ is of the form $y\eta^i$ for some $y \in X[n]$ and in this case we can find the (uniquely determined) y explicitely.

It is clear that Eilenberg-Zilber Lemma holds constructively for decidable X. If $u : X[n]$ then there exists exactly one surjective map $\eta : [n] \to [m]$ and one non degenerate element $v : X[m]$ such that $u = v\eta : X[n]$.

2 Model of type theory

A model is given by a collection of *contexts*. If Γ, Δ are context we have a collection $\Delta \to \Gamma$ of *substitutions* from Δ to Γ. This should form a category: we have a substitution $1 : \Gamma \to \Gamma$ and a composition operator $\sigma\delta : \Theta \to \Gamma$ if $\delta : \Theta \to \Delta$ and $\sigma : \Delta \to \Gamma$. Furthermore we should have $\sigma 1 = 1\sigma = \sigma$ and $(\theta\sigma)\delta = \theta(\sigma\delta)$. If Γ is a context we have a collection of *types over* Γ. We write $\Gamma \vdash A$ to express that A is a type over Γ. If $\Gamma \vdash A$ and $\sigma : \Delta \to \Gamma$ we should have $\Delta \vdash A\sigma$. Furthermore $A1 = A$ and $(A\sigma)\delta = A(\sigma\delta)$. If $\Gamma \vdash A$ we have also a colection of *elements of type* A. We write $\Gamma \vdash a : A$ to express that a is an element of type A. If $\Gamma \vdash a : A$ and $\sigma : \Delta \to \Gamma$ we should have $\Delta \vdash a\sigma : A\sigma$. Furthermore $a1 = a$ and $(a\sigma)\delta = a(\sigma\delta)$.

We have a *context extension operation*: if $\Gamma \vdash A$ then we have a new context $\Gamma.A$. Furthermore there is a projection $\mathsf{p} \in \Gamma.A \to \Gamma$ and a special element $\Gamma.A \vdash \mathsf{q} : A\mathsf{p}$. If $\sigma : \Delta \to \Gamma$ and $\Gamma \vdash A$ and $\Delta \vdash a : A\sigma$ we have an extension operation $(\sigma, a) : \Delta \to \Gamma.A$. We should have $\mathsf{p}(\sigma, a) = \sigma$ and $\mathsf{q}(\sigma, a) = a$ and $(\sigma, a)\delta = (\sigma\delta, a\delta)$ and $(\mathsf{p}, \mathsf{q}) = 1$.

If $\Gamma \vdash a : A$ we write $[a] = (1, a) : \Gamma \to \Gamma.A$. Thus if $\Gamma.A \vdash B$ and $\Gamma \vdash a : A$ we have $\Gamma \vdash B[a]$. If furtermore $\Gamma.A \vdash b : B$ we have $\Gamma \vdash b[a] : B[a]$. Models are usually presented by giving a class of special maps (fibrations), in our case they are the maps $\mathsf{p} : \Gamma.A \to \Gamma$, and the elements are the sections of these fibrations, in our case the maps $[a] : \Gamma \to \Gamma.A$ determined by an element $\Gamma \vdash a : A$.

We suppose furthermore one operation $\Pi\ A\ B$ such that $\Gamma \vdash \Pi\ A\ B$ if $\Gamma \vdash A$ and $\Gamma.A \vdash B$. We should have $(\Pi\ A\ B)\sigma = \Pi\ (A\sigma)\ (B\sigma^+)$ where $\sigma^+ = (\sigma\mathsf{p}, \mathsf{q})$. We have an abstraction operation λb such that $\Gamma \vdash \lambda b : \Pi\ A\ B$ if $\Gamma.A \vdash b : B$. We have an application operation such that $\Gamma \vdash \mathsf{app}(c, a) : B[a]$ if $\Gamma \vdash a : A$ and $\Gamma \vdash c : \Pi\ A\ B$. These operations should satisfy the equations

$$\mathsf{app}(\lambda b, a) = b[a], \qquad c = \lambda(\mathsf{app}\ c^+), \qquad (\lambda b)\sigma = \lambda(b\sigma^+), \qquad \mathsf{app}(c, a)\sigma = \mathsf{app}(c\sigma, a\sigma)$$

where we write $c^+ = (c\mathsf{p}, \mathsf{q})$ and $\sigma^+ = (\sigma\mathsf{p}, \mathsf{q})$.

To define a model of type theory with one universe, we assume that we have a special type $\Gamma \vdash U$ such that $U\sigma = U$ and $\Gamma \vdash A$ whenever $\Gamma \vdash A : U$. Furthermore we assume that $\Gamma \vdash \Pi\ A\ B : U$ whenever $\Gamma \vdash A : U$ and $\Gamma.A \vdash B : U$.

All equations we have been using can be grouped together in the equations of C*-monoid* [2]. There are the following equations of a monoid with a special constants $\mathsf{p}, \mathsf{q}, \mathsf{app}$ and operations (x, y) and λx

$$(xy)z = x(yz) \qquad\qquad x1 = 1x = x$$

$$\mathsf{p}(x, y) = x \qquad \mathsf{q}(x, y) = y \qquad (x, y)z = (xz, yz) \qquad 1 = (\mathsf{p}, \mathsf{q})$$
$$\mathsf{app}(\lambda x, y) = x[y] \qquad (\lambda x)y = \lambda(xy^+) \qquad 1 = \lambda\ \mathsf{app}$$

where we define $[y] = (1, y)$ and $x^+ = (x\mathsf{p}, \mathsf{q})$. We have $x^+(y, z) = (xy, z)$ and $x^+y^+ = (xy)^+$ and $x^+[y] = (x, y)$.

We can add also descibe a model of type theory with *dependent sums*. We should have $\Gamma \vdash \Sigma\ A\ B$ if $\Gamma \vdash A$ and $\Gamma.A \vdash B$. If $\sigma : \Delta \to \Gamma$ we should have $(\Sigma\ A\ B)\sigma = \Sigma\ (A\sigma)\ (B\sigma^+)$. If $\Gamma \vdash a : A$ and $\Gamma \vdash b : B[a]$ we should have $\Gamma \vdash (a, b) : \Sigma\ A\ B$. We require the equation $(a, b)\sigma = a\sigma, b\sigma$. We ask also for two operations $\Gamma \vdash \mathsf{p}c : A$ and $\Gamma \vdash \mathsf{q}c : B[\mathsf{p}c]$ if $\Gamma \vdash c : \Sigma\ A\ B$ and the equations $\mathsf{p}(a, b) = a$ and $\mathsf{q}(a, b) = b$.

2.1 Simplicial set model

If $\mathcal{C}$ is any small category, the presheaf model of type theory over $\mathcal{C}$ can be described as follows.

We write $X, Y, Z, \dots$ the objects of $\mathcal{C}$ and $f, g, h, \dots$ the maps of $\mathcal{C}$. If $f : X \to Y$ and $g : Y \to Z$ we write gf the composition of f and g. We write $1_X : X \to X$ or simply $1 : X \to X$ the identity map of X. Thus we have $(fg)h = f(gh)$ and $1f = f1 = f$.

A context is interpreted by a presheaf Γ: for any object X of $\mathcal{C}$ we have a set $\Gamma(X)$ and if $f : Y \to X$ we have a map $\rho \longmapsto \rho f,\ \Gamma(X) \to \Gamma(Y)$. This should satisfy $\rho 1 = \rho$ and $(\rho f)g = \rho(fg)$ for $f : Y \to X$ and $g : Z \to Y$.

A type $\Gamma \vdash A$ over Γ is given by a set $A\rho$ for each $\rho : \Gamma(X)$. Furthermore if $f : Y \to X$ we have $\rho f : \Gamma(Y)$ and we can consider the set $A\rho f$. We should have a map $u \longmapsto uf,\ A\rho \to A\rho f$ which should satisfy $u1 = u$ and $(uf)g = u(fg)$.

An element $\Gamma \vdash a : A$ is interpreted by a family $a\rho : A\rho$ such that $(a\rho)f = a(\rho f)$ for any $\rho : \Gamma(X)$ and $f : Y \to X$.

This can be seen as a concrete description of what is respectively a fibration and a section of this fibration.

If $\Gamma \vdash A$ we can define a new presheaf $\Gamma.A$ by taking $(\rho, u) : (\Gamma.A)(X)$ to mean $\rho : \Gamma(X)$ and $u : A\rho$. We define $(\rho, u)f = \rho f, uf$.

If we have a map $\sigma : \Delta \to \Gamma$ and $\Gamma \vdash A$ we define $\Delta \vdash A\sigma$ by $(A\sigma)\rho = A\sigma\rho$.

We can interpret dependent products $\Gamma \vdash \Pi\ A\ B$ and sums $\Gamma \vdash \Sigma\ A\ B$ if we have $\Gamma \vdash A$ and $\Gamma.A \vdash B$. For $\rho : \Gamma(X)$ we define $(u, v) : (\Sigma\ A\ B)\rho$ to mean $u : A\rho$ and $v : B(\rho, u)$. We define $(u, v)f = uf, vf$ for $f : Y \to X$. On the other hand an element of $(\Pi\ A\ B)\rho$ is a family w indexed by $h : Y \to X$ with

$$wh : \prod_{u : A\rho h} B(\rho h, u)$$

and such that $\mathsf{app}(wh, u)g = \mathsf{app}(whg, ug)$ if $h : Y \to X$ and $g : Z \to Y$. We define then $(wh)f = w(hf)$. We write $w = w1$.

We can interpret $\Gamma \vdash \lambda t : \Pi\ A\ B$ whenever $\Gamma.A \vdash t : B$ and $\Gamma \vdash \mathsf{app}(v, u) : B[u]$ if $\Gamma \vdash u : A$ and $\Gamma \vdash v : \Pi\ A\ B$. Here we write $[u]$ the map $\Gamma \to \Gamma.A$ defined by $[u]\rho = \rho, u\rho$. If $\rho : \Gamma(X)$ and $f : Y \to X$

we define $\mathsf{app}((\lambda t)\rho f, a) = t(\rho f, a) : B(\rho f, a)$ for $a : A\rho f$. We take $\mathsf{app}(v, u)\rho = \mathsf{app}(v\rho, u\rho) : B(\rho, u\rho)$. We can then check that we have

$$\mathsf{app}(\lambda t, u)\rho = t(\rho, u\rho) = t[u]\rho : B(\rho, u\rho)$$

if $\Gamma.A \vdash t : B$ and $\Gamma \vdash u : A$ and $\rho : \Gamma(X)$, which shows that the model validates the conversion rule $\Gamma \vdash \mathsf{app}(\lambda t, u) = t[u] : B[u]$.

For the simplicial model, we take $\mathcal{C}$ to be the cateory Δ, and we get a model of MLTT.

3 Kan simplicial set

We define Γ to be a Kan simplicial set if we have a filling operator which fills any horn $\Lambda_n^k \to \Gamma$ to $\Delta_n \to \Gamma$. We say that $\Gamma \vdash A$ is a Kan fibration iff given any $\alpha : \Delta_n \to \Gamma$, any partial section in $\prod_{i:\Lambda_n^k} A(\alpha\ i)$ can be extended to a section in $u : \prod_{i:\Delta_n} A(\alpha\ i)$.

We say that $\Gamma \vdash A$ has the *transfer property* iff we have two operators in $\prod_{\alpha:\Gamma^{\mathbf{I}}} A\alpha_0 \to A\alpha_1$ and $\prod_{\alpha:\Gamma^{\mathbf{I}}} A\alpha_1 \to A\alpha_0$ which sends the constant path to the identity map.

Lemma 3.1 *If $\Gamma \vdash A$ has the transfer property then given any $\alpha : \Gamma^{\mathbf{I}}$ and $i : \Delta_1$ and $u : A(\alpha\ i)$ we can find $v : \prod_{j:\Delta_1} A(\alpha\ j)$ such that $v\ i = u : A(\alpha\ i)$. It follows that given any $\alpha : \Delta_n \to \Gamma$ and $i : \Delta_n$ and $u : A(\alpha\ i)$ we can find $v : \prod_{j:\Delta_n} A(\alpha\ j)$ such that $v\ i = u : A(\alpha\ i)$.*

If $\Gamma \vdash A$ is a Kan fibration and $\Gamma, \Gamma.A$ are decidable then whenever we have $\alpha : \sigma_0 \to \sigma_1$ in $\Gamma[1]$ we can define a simplicial map $A_0 \to A_1$ where $A_i = A\sigma_i$. By the Kan property of $\Gamma \vdash A$ we first define a map on points $A_0[0] \to A_1[0]$. We can then extend this map using the Kan property of $\Gamma \vdash A$ to $f_\alpha : A_0[n] \to A_1[n]$ by induction on n and by case if the simplex $u : A_0[n]$ is degenerate or not. Furthermore we have a homotopy in $\Gamma.A$ between $u : A_0[n]$ and $f_\alpha u$. For $n = 2$ we have a prism, that we can fill by the Kan property getting the homotopy between u and $f_\alpha u$. There are different ways to fill this prism, and there does not seem to be a canonical way. What matters is that there is at least one way to do it.

This remark can be strengthened as follows. This next result is important since it is used both for showing that dependent product preserves the Kan property and for the interpretation of the elimination rule for the identity type.

Theorem 3.2 *If $\Gamma \vdash A$ is a Kan fibration and $\Gamma, \Gamma.A$ are decidable then $\Gamma \vdash A$ has the transfer property.*

Concretely, this corresponds to a program that given a path $(v_0, \dots, v_n)$ in $\Gamma([n+1])$ such that $v_i \epsilon^{i+1} = v_{i+1}\epsilon^{i+1}$ for $i < n$ and given t_0 in $Av_0\epsilon_0$ build an element t_1 in $Av_n\epsilon_{n+1}$.

Corollary 3.3 *If $\Gamma \vdash A$ and $\Gamma.A \vdash B$ are Kan fibration and $\Gamma, \Gamma.A$ are decidable then $\Gamma \vdash \Pi\ A\ B$ is a Kan fibration.*

Proof. We take $\alpha : \Delta_n \to \Gamma$ and $w(i) : (\Pi\ A\ B)\alpha(i)$ for $i : \Lambda_n^k$. We want to extend w to Δ_n. We take $i_0 : \Delta_n$ and $u : A\alpha(i_0)$. We can extend u to $\tilde{u}(i) : A\alpha(i)$ by Theorem 3.2 and Lemma 3.4. We then have the section $w(i)(\tilde{u}(i)) : B(\alpha, \tilde{u})(i)$ for $i : \Lambda_n^k$ that we can extend to $\tilde{v}(i) : B(\alpha, \tilde{u})(i)$ for $i : \Delta_n$. We define then

$$\mathsf{app}(\tilde{w}(i_0), u) = \tilde{v}(i_0) : B(\alpha, \tilde{u})(i_0) = B(\alpha(i_0), u)$$

and this is the required extension of w. Indeed if i_0 is in Λ_n^k we have

$$\mathsf{app}(\tilde{w}(i_0), u) = \tilde{v}(i_0) = \mathsf{app}(w(i_0), u) : B(\alpha(i_0), u)$$

for all $u : A\alpha(i_0)$ so that $\tilde{w}(i_0) = w(i_0) : (\Pi\ A\ B)\alpha(i_0)$ in this case and $\tilde{w}$ is an extension of w. □

If $\rho \in \Gamma$ we define $(\mathsf{Path}_A\ a\ b)\rho$ to be the set of elements w in $\mathbf{I} \to A\rho$ such that $w(0) = a\rho$ and $w(1) = b\rho$.

If $\Gamma \vdash a : A$ we can interpret $\Gamma \vdash \mathsf{Ref}\ a : \mathsf{Path}_A\ a\ a$ by taking $(\mathsf{Ref}\ a)\rho$ to be the constant path: it is the map $(\mathsf{Ref}\ a)\rho(i) = a\rho$ for $i : \mathbf{I}$.

The extensionality property of the path space holds for this model.

Lemma 3.4 *The path space satisfies the extensionality axiom: if we have $\Gamma.A \vdash b : B$ and $\Gamma.A \vdash c : B$ and $\Gamma.A \vdash p : \mathsf{Path}_B\ b\ c$ then we can find a section of $\Gamma \vdash \mathsf{Path}_{\Pi\ A\ B}\ (\lambda b)\ (\lambda c)$.*

Proof. Given ρ in Γ we have to define an object $\mathsf{ext}(p)\rho$ of type $(\mathsf{Path}_{\Pi\ A\ B}\ (\lambda b)\ (\lambda c))\rho$. This should be an element in $\mathbf{I} \to (\Pi\ A\ B)\rho$. We take $i : \mathbf{I}$ and we should define $\mathsf{ext}(p)\rho(i)$ in $(\Pi\ A\ B)\rho$. We define

$$\mathsf{ext}(p)\rho = \lambda i.\lambda u.p(\rho, u)(i)$$

for $i : \mathbf{I}$ and $u : A\rho$. We can then check that

$$\mathsf{app}(\mathsf{ext}(p)\rho(0), u) = p(\rho, u)(0) = b(\rho, u) : B(\rho, u)$$

and

$$\mathsf{app}(\mathsf{ext}(p)\rho(1), u) = p(\rho, u)(1) = c(\rho, u) : B(\rho, u)$$

Roughly speaking, what is going on is an isomorphism between function types of the form $\mathbf{I} \to (C \to D)$ and $C \to (\mathbf{I} \to D)$. □

We need the following notation: if $\Gamma \vdash a : A$ we write $\Gamma.A \vdash S(A, a)$ the “singleton type” associated to the element $\Gamma \vdash a : A$. If $\rho : \Gamma$ and $u : A\rho$ then $S(A, a)(\rho, u)$ is the set of paths $\omega : \mathbf{I} \to A\rho$ such that $\omega(0) = a\rho$ and $\omega(1) = u$. We have $S(A, a)[b] = \mathsf{Path}_A\ a\ b$ if $\Gamma \vdash b : A$.

Lemma 3.5 *Given $\Gamma \vdash a : A$ we define $\Gamma \vdash T$ by $T = \Sigma\ A\ S(A, a)$. We have $\Gamma \vdash (a, \mathsf{Ref}\ a) : T$. If $\Gamma \vdash A$ has the 2-transfer property then all elements in T are connected to $(a, \mathsf{Ref}\ a)$, i.e. there is a section of $\Gamma.T \vdash S(T, (a, \mathsf{Ref}\ a))$.*

Proof. Given $\rho : \Gamma$ and $(u, \omega) : T\rho$ we need to define an element of $\mathsf{Path}_{T\rho}\ (a\rho, (\mathsf{Ref}\ a)\rho)\ (u, \omega)$. For this, we use the square

$$\begin{tikzcd}
a\rho \arrow[r] \arrow[d] & a\rho \arrow[d, "\omega"] \\
a\rho \arrow[r, "\omega"'] & u
\end{tikzcd}$$

which is of type $\mathsf{Path}_{T\rho}\ (a\rho, (\mathsf{Ref}\ a)\rho)\ (u, \omega)$. □

Lemma 3.6 *If $\Gamma \vdash A$ is decidable and Kan and $\Gamma \vdash a_0 : A$, $\Gamma \vdash a_1 : A$ then $\Gamma \vdash \mathsf{Path}_A\ a_0\ a_1$ is decidable and Kan.*

Using Lemmas 3.5 and 3.6 and Theorem 3.2, we get a model of WMLTT with identity types. Lemma 3.4 shows that this model satisfies the extensionality axiom.

4 Propositional Reflection

Given a *decidable* Kan simplicial set A we define A^* which is a proposition. An element of $A^*[n]$ is of the form $\mathsf{S}_\eta(u)$ where $\eta : [n] \to [m]$ is a surjection and $u : A[0]^{m+1}$. Since A is decidable we have a canonical map $A \to A^*$. Also if A is a proposition we have a map $A^* \to A$. This is because it is possible to build for any sequence $a_0, \ldots, a_n$ in $A[0]^{n+1}$ an element u in $A[n]$ such that $u\epsilon^i = a_i$ where $\epsilon^i : [0] \to [n]$, $\epsilon^i 0 = i$. This is built by induction on n using the fact that A is a proposition and a Kan simplicial set.

5 Univalence

We define the universe by taking $U[n]$ to be the set of all small decidable Kan fibration $\Delta_n \vdash A$.

5.1 Any weak equivalence defines a path between types

Given a map $\sigma : A \to B$ between two decidable simplicial sets, we build a small decidable fibration $\mathbf{I} \vdash E$ such that $E0 = A$ and $E1 = B$. The set $\mathbf{I}[n]$ has $n+2$ elements $0 < \rho_1 < \ldots < \rho_n < 1$. We define $E0 = A[n],\ E1 = B[n]$ and $E\rho_k$ is the set of pairs a, b with $b : B[n]$ and $a : A[n-k]$ and $\sigma a = b\epsilon$ where $\epsilon : [n-k] \to [n]$ is the inclusion map.

We use the following characterisation of weak equivalence: given any commutative square

$$\begin{tikzcd} \partial\Delta_n \arrow[r, "a"] \arrow[d] & A \arrow[d, "\sigma"] \\ \Delta_n \arrow[r, "b"] & B \end{tikzcd}$$

we can find a map $u : \Delta_n \to A$ such that the upper triangle commutes and the lower triangle commutes up to homotopy relative to $\partial\Delta_n$, i.e. $\sigma u \sim b$ using the notation in [4].

Lemma 5.1 *If A is a Kan simplicial set and $a : A[n]$ and we have $a\epsilon^i \sim u$ then we can find $a' : A[n]$ such that $a'\epsilon^i = u$ and $a'\epsilon^j = a\epsilon^j$ for $i \neq j$.*

We show that $\mathbf{I} \vdash E$ is a Kan fibration if σ is a weak equivalence. Given an element $\rho : \Delta_n \to \mathbf{I}$ and a partial section $u\ i : E\ (\rho\ i)$ for $i : \Lambda^k_n$, we should build an element of $E\rho$. Since B is a Kan simplicial set, only the case $\rho = \rho_1$ is problematic. But this follows directly from the previous characterisation of weak equivalence. For instance in the case $n = 2$, we give two lines $\sigma a_0 \to b,\ \sigma a_1 \to b$ that we have to complete to a triangle. Since B has the Kan property, we can complete it to a triangle in B. We then have a line $\mu : \sigma a_0 \to \sigma a_1$ in B. Since σ is a weak equivalence we can find $\lambda : a_0 \to a_1$ in A such that $\sigma\lambda \sim \mu$, and we can replace μ by $\sigma\lambda$ using Lemma 5.1.

5.2 The universe is Kan

We describe the case $n = 2$ and define only the composition. Given for instance two fibrations $\Delta_1 \vdash E$ and $\Delta_1 \vdash F$ such that $E0 = A,\ E1 = B = F0,\ F1 = C$ we have to build a fibration $\Delta_1 \vdash G$ such that $G0 = A,\ G1 = C$. Given $\Delta_1 \vdash E$ we associate a transfer function $\alpha : A \to B$. There also, there does not seem to be a canonical definition of G. We have to define a set $G\rho$ for each $\rho : \Delta_1[n]$. The set $\Delta_1[n]$ is a linear poset with $n+2$ elements $0 < \rho_1 < \ldots < \rho_n < 1$. We define $G\rho_k$ to be the set of pairs (a, b) with a in $A[n-k]$ and b in $F\rho_k$ such that $b\epsilon = \alpha a$ where ϵ is the inclusion map $[n-k] \to [n]$.

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