Interaction Sequences

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Introduction

We present an abstract version of the notion of cuts between proofs. This leads to an argument of normalisation based on an analysis of what happens during the process of cut-elimination (and not on an induction on the complexity of the cut formula).

This paper is mathematically self-contained. A knowledge of infinitary propositional calculus, as presented in [6], may be useful for reading section 6.

1 Motivations

1.1 General Remarks

The idea of identifying a proof with a winning strategy for a game seems to come from Lorenzen [4, 2][1]. This identification is especially clear if we consider intuitionistic provability of arithmetical prenex formulae. For example, the game defined by a formula

$$\exists x.\forall y.\exists z.A(x, y, z),$$

where $A(x, y, z)$ is decidable, is that a player chooses a value for x, the opponent a value for y and then the player chooses a value for z. The player wins iff the formula $A(x, y, z)$ becomes true for this choice of values for x, y, z. In this case, it is clear that a winning strategy for this game corresponds exactly to an intuitionistic proof of the above formula.

Looking at examples of prenex formulae that are classically valid, such as $\exists x\forall y\ [f(x) \leq f(y)]$, it seems natural to try to extend this analogy between

[1] The author was lead to this identification by reading [1]

proofs and winning strategy in the case of classical logic by allowing the proof, when it has to make a move, to answer to any previous move of its opponent, or to play a new initial move. One can then hope to identify classical proofs with winning strategy for such games. This was suggested by Lorenz, then a student of Lorenzen [2].

To take into account connectives, we move directly to infinitary propositional calculus (see for instance [6]) and consider universal quantification as an infinite conjunction, and existential quantification as an infinite disjunction. Negation is not a primitive symbol, and is defined by the usual de Morgan rules.

Another idea, that comes from concurrency theory [3], is to interpret a strategy as an interactive programs and modus ponens as internal communication: given a winning strategy for $A \Rightarrow B$, which is classicaly defined as $\neg A \vee B$, and a winning strategy for A, one hopes to get a winning strategy for the game corresponding to B by letting the strategy for $A \Rightarrow B$ play against the strategy for A whenever its play concerns A. One expects then that the result of cut-elimination will be replaced by a proof showing that "internal chatters" end eventually.

When trying to put these ideas together, the difficulty is in the exact definition of what it means to "let two strategies play against each other". Trying to precise this leads to the notion of interaction sequence, which is a purely combinatorial notion.

One surprise is then that the main concepts about proofs, like the one of normal proofs, can be lifted at the level of interaction sequence. Basic facts about proofs, like cut-elimination, can also be expressed and proved at the level of interaction sequences.

We first present the notion of interaction sequence, and some of its basic properties. These are directly applied to a definition of classical provability for infinitary propositional formulae [6], for which modus ponens can be interpreted by internal communication. At the end, we present a concrete example of such a communication between proofs.

1.2 Introduction to interaction sequences

In order to motivate the introduction of interaction sequences, and our game-theoretical description of classical provability, we think it may be helpful to present briefly the calculus of Novikoff [5], which seems to be the first place in print where it is explicitely noticed that a Σ^1_0 formula provable classically has an intuitionistic proof.

The formulae of this system are of two sorts, existential and universal, and are inductively defined:

- if (A_i) is a family of universal formulae, then ΣA_i is an existential formula,
- if (A_i) is a family of existential formulae, then ΠA_i is an universal formula.

In these clauses, we restrict the index sets to be countable. Furthermore, Π and Σ are taken to be associative, commutative operations. Two formulae that differs by renaming on the indexes are identified. In particular, we can define the binary sum of two formulae (there are four cases according to the kind of formulae we add, but we get always an existential formula as a result). Negation is defined by the usual de Morgan rules. Notice that the true formula is the universal formula over the empty family of exsitential formulae, and the false formula is the existential formula over the empty family of universal formulae.

Notice that we can think of any such formula directly as a tree, with a "polarity", that says whether or not this tree is an existential or universal formulae. In our game-theoretic treatment of classical logic later, we simply introduce one connective, a generalised Sheffer connectives, that identifies a formula with a tree.

It is quite straightfoward to represent any formulae of Peano arithmetic in this infinitary propositional calculus. We do not do it in detail here (see the example at the end), but refer to Tait's or Novikoff's paper [5, 6].

The notion of provable formulae is defined inductively as follows:

- ΠA_i is provable iff each A_i is provable,
- ΣA_i is provable iff there exists i_0 such that, for all j, the formula $\Sigma_{i\neq i_0} A_i + A_{i_0 j}$ is provable.

Except for the non redondancy condition $i \neq i_0$, we can recognize in this definition the usual definition of cut-free provability in sequent calculus. We will come back later to this non redondancy condition.

The idea is now to notice that this inductive definition has a natural game theoretic interpretation. A formula can be thought canonically of as specifying a perfect information game between two players I and II (the polarity tells which player starts). Let us fix things, and say that player I starts playing for an existential formula. The inductive definition of the provability of the game A can be seen as describing a winning strategy for player I in a (rather unfair) game where the player I can at any point change his mind about a previous play, but also at any point resume a game where he has left it. Let us call this game the game $CF(A)$, because it corresponds to the notion of cut-free provability.

The principal result of Novikoff's paper is an intuitionistic proof that, if both $A + B$ and A are provable, then B is provable. In particular, we should be able, from a winning strategy q for $CF(A + B)$ and a winning strategy p for $CF(A)$, to compute a winning strategy for $CF(B)$.

It turns out, and that is one of the main point in this approch, that in term of strategies, there is a quite natural procedure to do this computation. I will describe it only in order to compute the first move in B, assuming, in order to simplify things, that B is existential and A universal (so that A is existential). The strategy p is waiting for a move of II. The strategy q gives a move for I in $A + B$. There are two cases.

If this move is in the B part, we get the first move for the resulting strategy, and the computation is finished.

If this move is in the A part, this can be considered as a move for II in A that is transmitted to p. The strategy p gives then a move that can be interpreted as an answer to the move of q, and we examine what q plays next. There is again two cases. If this move is in the B part, we get the first move for the resulting strategy, and the computation is finished, and so on.

We see that this computation is almost forced in term of strategies. Notice that in particular, we expect from Novikoff's result that this computation cannot go on forever, that is, eventually, the strategy q will play in the B part. We will show a direct combinatorial proof of this fact in proposition 1 (which is our main combinatorial result).

In order to get this result, we give a direct combinatorial characterisation of the kind of sequence moves that we get when we let such strategies p and q

interact. We retain only as information to what previous moves the strategy is answering, and this is surprisingly enough in carrying out the termination argument. The characterisation of the sequences we can get is captured by the notion of interaction sequences.

Finally, we must say a word about the non redondancy condition. The definition of cut-free provability that we shall later consider will be the following

- ΠA_i is provable iff each A_i is provable,
- ΣA_i is provable iff there exists i_0 such that, for all j, the formula $\Sigma A_i + A_{i_0 j}$ is provable.

This is the same as Novikoff's definition, except that we allow the strategy to play twice (or more) the same move. This looks clearly strange in term of strategies for games, but it corresponds more closely to what is done in sequent calculus (see [6] for instance). Also, it should be pointed out, that even if we start with strategies that are not redondant, we produce naturally by cuts strategies that are redondant. It seems extremely likely that it is possible to complicate a little our present notion of cut as internal communication in order to eliminate this redondancy, but we have not yet complete results in this direction.

2 Interaction Sequences

2.1 Definition

An **interaction sequence** is a pair (V, f) such that $V(0)$ is empty, $V(1) = \{0\}$, $f(1) = 0$, the function f is defined on an initial segment $[1, N]$ and for $n < N$

$$V(n+1) = \{n\} \cup V(f(n)), \quad f(n+1) \in V(n+1).$$

If (V, f) is defined for all positive integers, and for all N, (V, f) is an interaction sequence on $[1, N]$, we say that (V, f) is an **infinite interaction sequence**.

Notice that, if (V, f) is an interaction sequence, we always have $f(n) < n$ and $f(n), n$ are of distinct parity. Furthemore, V is uniquely determined by f and we must have $f(1) = 0$ and $f(2) = 1$. There is a choice for the next value $f(3)$ that may be 0 or 2, and if $f(3) = 0$, we must have $f(4) = 3$.

It may help the intuition of some readers to think of such an interaction sequence (V, f) to be built progressively in "stages": at the stage $n+1$, the choice of $f(n+1)$ is limited and has to be in the set $\{n\} \cup V(f(n))$.

For a motivation of the notion of interaction sequence, see the example in the last section. This notion comes naturally when one tries to analyse the interaction of two cut-free proofs in infinitary classical propositional logic. Here it is defined abstractly, without any references to proofs, and the main properties of interaction sequences can be seen as abstract formulation of the corresponding notion on proofs (the proposition 2 below corresponds to cut-elimination).

We let $y \prec x$ mean that $x \in f(V(y))$. By a direct induction on y, $y \prec x$ iff there exists a sequence $y_1, \ldots, y_n$ such that $y_1 = f(y-1)$, $y_{k+1} = f(y_k - 1)$ and $y_n = x$. Hence $\prec$ is transitive.

Lemma 1 *If $y \prec x$, then $V(x)$ is a strict initial segment of $V(y)$.*

Proof: By the alternative definition of $\prec$. $\square$

We shall need a slight generalisation of the notion of interaction sequence. If $A = \{n_0, \ldots, n_k\}$, with $n_0 < \ldots < n_k$ and f is a function defined at least on $\{n_1, \ldots, n_k\}$, we say that f **defines an interaction** on A iff there exists an interaction sequence (V, g) defined on $[1, k]$ such that $f(n_p) = n_{g(p)}$ for $p = 1, \ldots, k$.

If $p = g(i)$, $q = g(j)$, we write $q \prec p$ (f, A) for the fact that $j \prec i$ relatively to the interaction sequence (V, g).

It can be seen directly that the following algorithm checks whether or not a function f defines an interaction on $\{n_0, \ldots, n_k\}$.

- If $k = 0$, then f does define an interaction on $\{n_0\}$.
- If $k > 0$, check recursively whether or not f defines an interaction on the set $\{n_0, \ldots, n_{k-1}\}$:
  - if not, then f does not define an interaction on $\{n_0, \ldots, n_k\}$,
  - if yes, we know that $f(n_{k-1})$ is of the form n_p, with $p < k-1$. If furthermore $f(n_k) = n_{k-1}$, then f defines an interaction on $\{n_0, \ldots, n_k\}$. Otherwise, f defines an interaction on $\{n_0, \ldots, n_k\}$ iff $f(n_k) \in \{n_0, \ldots, n_{p-1}\}$.

Lemma 2 *If f defines an interaction on $\{n_0, \ldots, n_r\}$, $f(n_q) = n_p$ and n_q is not in the set $f(\{n_{q+1}, \ldots, n_r\})$, then f defines an interaction on the set $\{n_0, \ldots, n_{p-1}, n_{q+1}, \ldots, n_r\}$.*

Proof: By induction on $r - q$, using the previous algorithm. $\Box$

If A is an infinite subset $\{n_0, n_1, \ldots\}$, and f is a function defined at least on A, we say that f **defines an interaction** on A iff f defines an interaction on each $\{n_0, \ldots, n_k\}$.

Let us define $\mathsf{depth}(f, 0) = 0$, $\mathsf{depth}(f, n) = \mathsf{depth}(f, f(n)) + 1$ for $n > 0$. The integer $\mathsf{depth}(f, n)$ is called the **depth** of n for f. We say that (V, f) is of **bounded depth** iff there exists N such that $\mathsf{depth}(f, n) < N$ for all n.

The following definitions will not be needed in the next two sections, but are needed for the definition of classical provability. We say that an interaction sequence f is **cut-free** iff $f(2p) = 2p - 1$ whenever $2p$ is in the domain of f. More generally, if f defines an interaction on $\{n_0, \ldots, n_k\}$, we say that f is cut-free iff $f(n_{2p}) = n_{2p-1}$ whenever $2p \leq k$.

Lemma 3 *If (V, f) is an interaction sequence and $0 < n$, then f defines an interaction on $V(n) \cup f(V(n))$. If n is odd, f defines a cut-free interaction on $V(n) \cup f(V(n))$.*

Proof: Let $C(n)$ be the set $V(n) \cup f(V(n))$. If $k \in V(n)$, then $n \prec f(k)$ and hence, by lemma 1, $V(f(k)) \subset V(n)$, so that $f^2(k) \in V(n)$. This shows that $f(C(n)) \subset C(n)$.

The set $C(n)$ can be described as the set $\{n_1, f(n_1), n_2, f(n_2), \ldots\}$ with $n_1 = n - 1$ and $n_{k+1} = f(n_k) - 1$. By induction on n, applying the algorithm described above, we show that f defines an interaction on $C(n)$. Indeed, by what is just shown, we have $f^2(n_1) = n_k \in V(n)$, and so $n_k = n_2$ or $n_k < f(n_2)$. This shows by induction that f defines an interaction on $\{f(n_1), n_2, f(n_2), \ldots\} = \{f(n_1)\} \cup C(f(n_1))$. It is then direct that f defines an interaction on $C(n) = \{n_1, f(n_1)\} \cup C(f(n_1))$.

Let $C(n)$ be $\{m_0, m_1, \ldots, m_l\}$. We have by construction $f(m_{l-2k}) = m_{l-2k-1}$ whenever $2k < l$. Hence f defines a cut-free interaction on $C(n)$ if l is even, which holds iff n is odd. $\Box$

Finally, we define inductively $\mathsf{index}(f, n)$ for n in the domain of f by

- $\mathsf{index}(f, n) = n$ if $f(n) = 0$,
- otherwise, $\mathsf{index}(f, n) = \mathsf{index}(f, f(n))$.

2.2 Main Properties

In this section, we suppose given an infinite interaction sequence (V, f).

Lemma 4 *if* $f(x) > 0$, *then* $x \prec f(f(x))$.

Proof: We have $f(x) \in V(x)$, hence $f(f(x)) \in f(V(x))$. $\Box$

If $A \subseteq \mathbf{N}$, $S_A(x)$ denotes $A \cap V(x)$.

An infinite subset $A = \{n_k\}$ is called **good** iff $f(A) \subseteq A$ and $S_A(n_{k+1}) = \{n_k\} \cup S_A(f(n_k))$.

Notice that $A = \mathbf{N}$ is good. Also, if A is good, then f defines an interaction on A.

Lemma 5 *If* $A = \{n_k\}$ *is good, either for all* q *there exists* $r > q$ *such that* $n_q = f(n_r)$, *or there exists a good subset* $\{m_l\}$ *and* p *such that* $n_i = m_i$ *for* $i < p$ *and* $m_p \prec n_p$.

Proof: If A is good, n_q not in $f(A)$, and $n_p = f(n_q)$, let (m_l) be defined by $m_i = n_i$ for $i < p$, and $m_{p+i} = n_{q+1+i}$. It is clear that (m_l) is strictly increasing. Let $B = \{m_l\}$. Lemma 2 shows that $f(B) \subseteq B$. Furthermore

$$S_A(n_{q+1}) = \{n_q\} \cup S_A(n_p) = \{n_q, n_{p-1}\} \cup S_A(f(n_{p-1}))$$

and hence

$$S_B(m_p) = \{m_{p-1}\} \cup S_B(f(m_{p-1})).$$

It is direct also that $S_B(m_{i+1}) = \{m_i\} \cup S_B(f(m_i))$ if $i + 1 < p$ or $p < i + 1$.

It follows that B is good.

Notice also that $m_p \prec n_p$ because $n_q \in V(n_{q+1})$. $\Box$

Proposition 1 *Given an infinite interaction sequence* (V, f), *there exists an infinite sequence* $u_1 < u_2 < u_3 \ldots$ *such that* $f(u_{p+1} - 1) = u_p$ *for all* p.

Proof: This can be reformulated by saying that $\prec$ is not well-founded. Were $\prec$ well-founded, we could find a good subset $\{n_k\}$ such that n_{k+1} is $\prec$-minimal for good subsets starting with $n_0, ..., n_k$. By lemma 5, we have that for all p, there exists $q > p$ such that $n_p = f(n_q)$, and we get a contradiction by lemma 4. $\Box$

In the important special case of bounded depth sequences, we can build effectively a sequence (u_p) such that $u_{p+1} \prec u_p$. The algorithm is built by induction on a bound N of the depth. If $\mathsf{depth}(f, n)$ is always $< N$, we apply the induction hypothesis. Otherwise, lemma 2 shows that two segments of the form $[f(n), n]$ with $\mathsf{depth}(f, n) = N$ are such that they are disjoint or one is strictly included into another. We progressively remove all these segments that are maximal. In this way, either we are left with an infinite subset, which is a good subset $\{n_k\}$ where all $\mathsf{depth}(f, n_k)$ are $< N$, and we apply the induction hypothesis, or we are left with a finite subset, and the left extremity of the segments form a sequence (u_p) such that $u_{p+1} \prec u_p$ for all p.

2.3 Cut-elimination for interaction sequences

An infinite interaction sequence (V, f) is said to be **winning** iff $\prec$ is well-founded over odd integers. If $A \subseteq N$ is infinite, we define in a corresponding way when f defines a winning interaction on A.

Lemma 6 *If (V, f) is an interaction sequence on $[1, n_k]$ and $\{n_0, \ldots, n_k\}$ is a set X such that $f(n_j) \in X$ for $j = 1, \ldots, k$ and $f(n) \in \{n_1, \ldots, n_k\}$ implies $n \in X$, then f defines an interaction sequence on X.*

Proof: By induction on k.

If $k = 1$, then we have $f(n_1) = n_0$ and hence f defines an interaction on $\{n_0, n_1\}$.

If $1 < k$, and the lemma holds for all $p < k$, let (V, f) and X satisfying the hypothesis of the lemma. By induction hypothesis, f defines an interaction on $\{n_0, \ldots, n_{k-1}\}$.

If $f(n_k) = n_{k-1}$ then f defines an interaction on $\{n_0, \ldots, n_k\}$.

Otherwise, we have $f(i) \neq n_{k-1}$ for $i \in]n_{k-1}, n_k]$, and hence, by lemma 2, if we let n_p be $f(n_{k-1})$ we have $f(n_k) < n_p$. The hypothesis of lemma 6 apply then to the set $\{n_0, \ldots, n_{p-1}, n_k\}$ and hence f defines an interaction on this set. This implies that f defines an interaction on $\{n_0, \ldots, n_k\}$. $\square$

Lemma 6 extends directly to the case of an infinite set $X = \{n_0, n_1, \ldots\}$ such that $f(n_j) \in X$ for $j = 1, \ldots$ and $f(n) \in \{n_1, \ldots\}$ implies $n \in X$.

We suppose given an interaction sequence (V, f).

Let $I \subset N$ be the set of integers i such that $f(i) = 0$. If $i \in I$, let A_i be the set of integers n such that $\mathsf{index}(f, n) = i$. The set A_i satisfies the two conditions of lemma 6, and so f defines an interaction sequence on A_i.

Lemma 7 *If $i \in I$, and n is even, then $n \in A_i$ iff i is the least element of $V(n)$. If n is odd and $n \in A_i$, then $n + 1 \in A_i$.*

Proof: First, it is clear that i is odd, and that $i+1 \in A_i$. Let $n > 0$ be even. The least integer k such that $f^k(n) = 0$ is even. Let $i = f^{k-1}(n) = \mathsf{index}(f, n)$. By lemma 1 and lemma 4, $V(f^{k-2}(n))$ is an initial segment of $V(n)$. Hence we are reduced to prove that if $f(n) = i$ and $f(i) = 0$, then i is the least element of $V(n)$. This is clear if $n = i+1$. If we are in the case $i \in V(f(n-1))$ and hence $i < f(n-1)$, then, using lemma 2, we can remove the segment $[f(n-1), n-1]$ until we get to the case $n = i+1$.

This proves that if n is even, then $n \in A_i$ iff i is the least element of $V(n)$.

If $n > i$ is odd, and $n \in A_i$, then $f(n) \in A_i$ and $f(n)$ is even, i is the least element of $V(f(n))$. Hence i is the least element of $V(n+1)$ and $n+1 \in A_i$. $\square$

Corollary 1 *If $i \in I$, and n is even, $m \prec n$, and $n \in A_i$, then $m \in A_i$, and $m \prec n\ (f, A_i)$.*

If $J \subseteq I$ and X_J denotes the complement of the union of all sets A_i for $i \in J$, then X_J satisfies the two conditions of lemma 6, and so f defines an interaction sequence on X_J.

Proposition 2 (cut-elimination) *Let $J \subseteq I$ be such that f defines a winning interaction sequence on each infinite A_i for $i \in J$. If (V, f) is a winning interaction sequence, then f defines a winning interaction on X_J.*

Proof: Proposition 1 and the corollary of lemma 7 show that X_J is infinite, because otherwise, $\prec$ will be well-founded both on odd and even integers.

If f does not define a winning interaction on X_J, then there exists two infinite increasing sequences (x_k) and (y_k) in X_J such that $f(y_k) = x_k$, and x_{k+1} is the next element coming after y_k in X_J, and all x_k are odds.

For each k, we show by induction on $l \leq k$ that f defines an interaction on $Y_l = [0, x_{k+1}[\backslash \cup_{i<l} [x_{k-i}, y_{k-i}]$. Indeed, we have $f(p) \neq y_{k-l}$ for $p \in Y_l$ and $y_{k-l} < p$. Hence, by lemma 2, if f defines an interaction on Y_l, for $l < k$, then it defines an interaction on Y_{l+1}.

It follows that f defines an interaction on $Y = [0, x_1[\cup \bigcup]y_k, x_{k+1}[$. This set Y is infinite, because f is winning. Since $f(y_k) = x_k$ for all k, we have that $n \prec m\ (f, Y)$ implies $n \prec m$. It follows that $\prec\ (f, Y)$ is well-founded on odd integers. Since $X_J \cap Y \subseteq [0, x_1[$ is finite, the corollary of lemma 7 shows that $\prec\ (f, Y)$ is also well-founded on even integers. We get then a contradiction from proposition 1. $\square$

3 Games

We use capital letters $A, B, S, \ldots$ for denoting finite sequences (or words). We denote by Sx the concatenation of S and x, and $<>$ denotes the empty sequence. If $S = x_1 \ldots x_n$, then n is the **length** of S. We say that a sequence T **extends** the sequence S iff T is of the form $Sx_1 \ldots x_p$.

All the objects we consider here, games and strategies, are considered given intuitionistically. In particular, they are computable objects.

3.1 Games and Strategies

A **game** G is a set of sequences which is such that $<> \in G$ and $S \in G$ whenever $Sx \in G$. The elements of G are called **game history**. If $S \in G$, the set $M_G(S) = \{x \mid Sx \in G\}$ is called the set of **possible moves from** S**.**

A **strategy** is a function ϕ defined on some elements of G of even length, and such that $\phi(S) \in M_G(S)$ whenever $\phi(S)$ is defined. The strategy is exactly defined on elements of G of even length that **follow the strategy** ϕ, where $s_1 \ldots s_n$ follows the strategy ϕ iff $\phi(s_1 \ldots s_{2k})$ is defined and is s_{2k+1} for all k such that $2k < n$.

Given a strategy ϕ, we say that an infinite sequence $s_1 s_2 \ldots$ **follows the strategy** ϕ iff $s_1 \ldots s_n$ follow the strategy ϕ for all n.

3.2 Debate associated to a game

Let f be an interaction on $[1, n]$ and S a sequence $x_1 \ldots x_n$ of length n, we define for each $k \leq n$ a sequence $I(f, S, k)$ of length $\mathsf{depth}(f, k)$ by

- $I(f, S, 0) = <>$,
- $I(f, S, k)$ is the concatenation $I(f, S, f(k))x_k$ if $m > 0$.

Given a game G we let G^* be the set of sequences $(f(1), s_1) \ldots (f(n), s_n)$ such that

- f is an interaction on $[1, n]$ and
- for all $k \leq n$ we have $I(f, s_1 \ldots s_n, k) \in G$.

It is direct that this defines a game, called the **debate associated** to the game G.

We say that a strategy for G^* is **winning** iff for any infinite sequence $(f(1), s_1)(f(2), s_2)\ldots$, that follows this strategy, the infinite interaction sequence f is winning.

It may help the intuition of the reader to think about what happens during a real debate on a given topic between two persons. Both defend arguments, can change for a while their position, but also, at any point, can resume the debate at a point it was left before. This is what the game G^* represents, where G can be said to represent the "topic" of the debate.

3.3 Cut-Free Strategy

If G is a game, an element $(f(1), s_1)\ldots(f(n), s_n) \in G^*$ is **cut-free** iff f is cut-free.

A **cut-free strategy** for a game G^* is a function ϕ defined on some elements of G^* of even length that are cut-free. Such a strategy ϕ is defined exactly on sequences that **follow the strategy** ϕ and the sequence $(f(1), s_1)\ldots(f(n), s_n)$ follows the strategy ϕ iff f is cut-free and $(f(p+1), s_{p+1})$ is equal to $\phi((f(1), s_1)\ldots(f(p), s_p))$ for all even $p < n$.

It is clear that any strategy for G^* defines a cut-free strategy by restriction.

Intuitively, a cut-free strategy tells how to behave in a debate against an opponent that never changes in mind.

We recall that, if f is an interaction sequence on $[1, n]$, we have written $V(n+1)$ the set inductively defined $\{n\} \cup V(f(n))$.

If $S = (f(1), s_1)\ldots(f(n), s_n) \in G^*$ is of even length, we know by lemma 3 that f defines a cut-free interaction on the set $V(n+1) \cup f(V(n+1)) = \{m_0, \ldots, m_l\}$, so that there exists an interaction g defined on $[1, l]$ such that $f(m_i) = m_{g(i)}$ for $i \leq l$. We let then $C(S)$ be the cut-free element $(g(1), s_{m_1})\ldots(g(l), s_{m_l}) \in G^*$, and $F(S)(i) = m_i$ for $i \leq l$.

Let ϕ be a cut-free strategy. We define a strategy $E(\phi)$ for G^* by computing $(q, s) = \phi(C(S))$ and letting $E(\phi)(S)$ be $(F(S)(q), s)$ for S of even length. The strategy $E(\phi)$ is called the **extension** of the cut-free strategy ϕ.

A cut-free strategy is said to be **winning** iff the relation of extension is well-founded on sequences that follow this strategy.

Lemma 8 *A winning strategy for G^* defines a winning cut-free strategy by restriction. Conversely, the extension of a winning cut-free strategy is a winning strategy.*

Proof: Direct from the definition. $\square$

4 Classical provability

4.1 Classical Formulae

The formulae are defined inductively by the unique rule:

- if $A_i,\ i \in I$ is a family of formulae, then $A = |(A_i, i \in I)$ is a formula.

Intuitively, $|$ is a generalised Scheffer connective, and A says that the formulae A_i are incompatible, i.e. A holds iff at least one A_i does not hold.

In particular, the formula $0 = |(A_i, i \in \emptyset)$, is false under this interpretation. We write $|A$ for $|(A)$ where (A) is a family with one formula A. It represents the negation of A. Thus the formula $1 = |0$ is true under this interpretation.

If $A = |(A_i, i \in I)$ is a formula, and K is a subset of I, we let $A(K)$ be the formula $|(A_i, i \in K)$.

This language is directly seen to be equivalent to infinitary propositional calculus as described in [6]. As shown in Tait's paper [6], this calculus contains naturally Peano arithmetic.

4.2 Classical Games

Each formula can be seen as a tree. To each formula A, we associate the game G_A where, intuitively, each player chooses alternatively a subtree of the tree already chosen by the opposite player. Formally, if $A = |(A_i, i \in I)$, then G_A is the set with the empty sequence and the sequences of the form iS, with $i \in I$ and $S \in G_{A_i}$.

We define a **proof** of A to be a winning strategy for the game G^*_A. We say that A is **provable** iff it has a proof.

Notice that the formula 0 is not provable with this definition. There is only one strategy for G^*_A if $A = 1$, and it is a winning strategy, so that $1 = |0$ is provable.

A winning cut-free strategy of G^*_A can directly be seen as a normal proof of A in the sense of Tait in [6] where we cannot have two consecutive rules of or-introduction.

4.3 Principal Properties

Let $A = |(A_i, i \in I)$ and K be a subset of I. If $S \in G^*_A$ is the sequence $(f(1), s_1) \dots (f(n), s_n)$, we say that a move $(f(p), s_p)$ **plays in** $A(K)$ iff $\mathsf{index}(f, p) = k$ is such that $s_k \in K$. Let $(f(p_1), s_{p_1}) \dots (f(p_l), s_{p_l})$ be the subsequence of S of elements $(f(p), s_p)$ that play in K. By lemma 6, there exists an interaction sequence g on $[1, l]$ such that $f(p_i) = p_{g(l)}$ for $i = 1, \dots, l$. We let $p_K(S) \in G^*_{A(K)}$ be the sequence $(g(1), s_{p_1}) \dots (g(l), s_{p_l})$.

If S is the sequence $(f(1), s_1) \dots (f(n), s_n)$, and $k \leq n$ is such that $f(k) = 0$, let $(0, s_k)(f(p_1), s_{p_1}) \dots (f(p_l), s_{p_l})$ be the subsequence of S of elements $(f(p), s_p)$ such that $\mathsf{index}(f, p) = k$. By lemma 6, there exists an interaction sequence g on $[1, l]$ such that $f(p_i) = p_{g(l)}$ for $i = 1, \dots, l$. We let $p_k(S) \in G^*_{A_{s_k}}$ be the sequence $(g(1), s_{p_1}) \dots (g(l), s_{p_l})$.

Proposition 3 (modus ponens) *If*

- $A = |(A_i, i \in I)$ *is provable,*
- $I = J \cup K$ *is a partition of* I,
- A_j *is provable for* $j \in J$,

then the formula $A(K) = |(A_i, i \in K)$ *is provable.*

Proof: Let ϕ be a winning strategy for A, and ϕ_j be a winning strategy for A_j, for $j \in J$. We say that a sequence $S \in G^*_A$ following ϕ is **correct** w.r.t. (ϕ_j) iff it is such that $p_k(S)$ follows ϕ_{s_k} whenever $f(k) = 0$ and $s_k \in J$.

Proposition 2 shows that the following extension $G(S)$ of S, for S sequence of even length $(f(1), s_1) \dots (f(n), s_n)$ following ϕ and correct w.r.t. (ϕ_j), is well defined:

- if $\phi(S) = (f(n+1), s_{n+1})$ and $\mathsf{index}(f, f(n+1)) = k$ is such that $s_k \in K$, then $G(S) = S(f(n+1), s_{n+1})$,

- otherwise, $\mathsf{index}(f, f(n+1)) = k$ is such that $s_k \in J$. We consider then the subsequence $(0, s_k)(f(p_1), s_{p_1}) \dots (f(p_l), s_{p_l})$ of $S(f(n+1), s_{n+1})$, built from elements $(f(p), s_p)$ such that $\mathsf{index}(f, p) = k$, and g such that $f(p_i) = p_{g(i)}$ for $i = 1, \dots, l$. Since $p_k(S)$ follows ϕ_k, the element

  $$\phi_k(p_k(S(f(n+1), s_{n+1}))) = \phi_k((g(1), s_{p_1}) \dots (g(l), s_{p_l})) = (m, s)$$

  is well defined. We let $G(S)$ be $G(S(f(n+1), s_{n+1})(p_m, s))$.

Notice that $G(S)$ is of odd length, extends S and its last move plays in K.

We can now define simultaneously by induction a strategy ψ for $A(K)$, and for any sequence S following ψ, a sequence $F(S)$ such that $F(S)$ follows ϕ, is correct w.r.t. (ϕ_j) and $p_K(F(S)) = S$. If S is of even length, let (p, s) be the last element of $G(F(S))$. There exists then a unique q such that $p_K(G(F(S)) = S(q, s)$ and we let $\psi(S)$ be (q, s) and $F(S\psi(S))$ be $G(F(S))$. If S is of odd length, and $S(p, s) \in G_{A(K)}$, we take $F(S(p, s))$ to be $F(S)(p, s)$. $\square$

The idea behind this proof is simply this: whenever ϕ is playing in the part indexed by K, we let the corresponding ϕ_j answering until we go outside the "K part". Proposition 2 shows that this will happen eventually.

Proposition 4 *(consistency) For any formula A, at least one formula A or $|A$ is not provable.*

Proof: Because 0 is not provable. This follows also directly from proposition 1. $\square$

It is clear that if $A = |(A_i, i \in I)$ and $K \subseteq I$ is such that $A(K)$ is provable, then A is provable, because a winning strategy for $A(K)$ is also a winning strategy for A.

Proposition 5 *If $A = |(A_i, i \in I)$ is provable, $K \subseteq I$ and there is an onto map $\rho: I \to K$ such that $A_{\rho(i)} = A_i$ for all $i \in I$ and $\rho(i) = i$ for $i \in K$, then $A(K)$ is provable.*

Proof: If $S \in G^*_A$ is the sequence $(f(1), s_1) \dots (f(n), s_n)$, let $G(S)$ be the sequence $(f(1), s'_1) \dots (f(n), s'_n)$, where $s'_i = \rho(s_i)$ if $f(i) = 0$, and $s'_i = s_i$ if $f(i) \neq 0$. It is clear that $G(S) \in G^*_{A(K)}$.

Let ϕ be a winning strategy of A. We define by simultaneous induction a strategy ψ for $A(K)$ and for any sequence S following ψ, a sequence $F(S)$ such that $F(S)$ follows ϕ and $G(F(S)) = S$.

If S is of even length, we compute $\phi(F(S)) = (p, s)$. If $p = 0$, we let $\psi(S)$ be $(p, \rho(s))$ and $F(S(p, s))$ be $F(S)(p, s)$. If $p \neq 0$, we let $\psi(S)$ be (p, s) and $F(S(p, s))$ be $F(S)(p, s)$.

If S if of odd length, and $S(p, s) \in G_{A(K)}$, we let $F(S(p, s))$ be $F(S)(p, s)$. $\square$

From proposition 3 and proposition 5 follows easily the equivalence of our notion of provable formulae with the usual definition of classical provability (as defined in [6]).

4.4 Example

A winning strategy can be seen as an interactive program, and proposition 3 interprets modus ponens as internal communication [3]. Here is an example of such a situation.

Given a function f on integer as a parameter, both formulae

$$A(f) = \forall x.\exists y \geq x.\forall z \geq x.\ [f(y) \leq f(z)]$$

and

$$B(f) = A(f) \Rightarrow \exists u_1, u_2, u_3.\ [u_1 < u_2 < u_3]\ \wedge\ [f(u_1) \leq f(u_2) \leq f(u_3)]$$

are provable. The second formula is even provable intuitionistically, but $A(f)$ holds only classically, if f is a parameter.

We will now define a winning cut-free strategy P for $A(f)$ and a winning cut-free strategy Q for $B(f)$. By lemma 8, this defines a winning strategy for $A(f)$ and $B(f)$ and proposition 3 leads then to a winning strategy for

$$\exists u_1, u_2, u_3.\ [u_1 < u_2 < u_3]\ \wedge\ [f(u_1) \leq f(u_2) \leq f(u_3)].$$

Such a winning strategy can be seen as a program computing u_1, u_2, u_3 such that $u_1 < u_2 < u_3$ and $f(u_1) \leq f(u_2) \leq f(u_3)$.

Rather than giving formally these winning cut-free strategy, we will explain them heuristically.

The winning cut-free strategy P for $A(f)$ can be described as follows:

- the opponent gives a value for $x = a$,
- P answers $y = a$,
- the opponent gives a value for $z = a_1$. If $f(a) \leq f(a_1)$, P has won.

- If $f(a) > f(a_1)$, P changes its mind and plays $y = a_1$ instead,
- the opponent gives a value for $z = a_2$. If $f(a_1) \leq f(a_2)$, P has won.
- If $f(a_1) > f(a_2)$, P changes its mind and plays $y = a_2$ ...

Since $\mathbf{N}$ is well-founded, P is going to win eventually.

Here is a description of Q seen as a cut-free strategy for the formula

$$\exists x.\forall y \geq x.\exists z \geq x.[f(y) > f(z)] \vee (\exists u_1, u_2, u_3)[u_1 < u_2 < u_3 \wedge f(u_1) \leq f(u_2) \leq f(u_3)].$$

This is described informally:

- Q chooses $x = 0$,
- the opponent chooses a value $y = a_1$,
- Q changes its mind and plays $x = a_1 + 1$,
- the opponent chooses a value $y = a_2$, such that $a_2 \geq a_1 + 1$,
- if $f(a_1) > f(a_2)$, Q resumes the game with its initial value 0 for x, and wins by playing $z = a_2$. If $f(a_1) \leq f(a_2)$, Q changes its mind and plays $x = a_2 + 1$,
- the opponent chooses a value $y = a_3$, such that $a_3 \geq a_2 + 1$,
- if $f(a_3) > f(a_2)$, Q resumes the game with the value $a_1 + 1$ for x, and wins by playing $z = a_3$. Otherwise, $f(a_1) \leq f(a_2) \leq f(a_3)$, and Q wins by playing $u_1 = a_1, u_2 = a_2, u_3 = a_3$.

We are going now to show an example of an interaction between these two proofs (identified with cut-free strategies), in the case where the values of f are given by

$$f(0) = 10,\ f(1) = 5,\ f(2) = 3,\ f(3) = 7,\ f(4) = 4,\ f(5) = 11,\ f(6) = 29, \ldots$$

Here are the moves, as they are given by proposition 3:

1. Q plays $x = 0$,
2. P plays $y = 0$,
3. Q changes its mind, plays $x = 1$,

4. P plays $y = 1$,
5. $f(0) > f(1)$, hence Q plays $z = 1$,
6. P plays $y = 1$,
7. Q plays $x = 2$,
8. P plays $y = 2$,
9. $f(1) > f(2)$, hence Q plays $z = 2$,
10. P plays $y = 2$,
11. Q plays $x = 3$,
12. P plays $y = 3$,
13. $f(3) \geq f(2)$, hence Q plays $x = 4$,
14. P plays $y = 4$,
15. $f(4) < f(3)$, hence Q plays $z = 4$,
16. P plays $y = 4$,
17. $f(4) \geq f(2)$, hence Q plays $x = 5$,
18. P plays $y = 5$,
19. $f(5) \geq f(4)$, hence Q plays $u_1 = 2,\ u_2 = 4,\ u_3 = 5$.

The interaction sequence g associated to this interaction is given by:

$$g(1) = 0,\ g(2) = 1,\ g(3) = 0,\ g(4) = 3,\ g(5) = 2,\ g(6) = 1,$$

$$g(7) = 0,\ g(8) = 7,\ g(9) = 6,\ g(10) = 1,\ g(11) = 0,\ g(12) = 11,$$

$$g(13) = 0,\ g(14) = 13,\ g(15) = 12,\ g(16) = 11, g(17) = 0,\ g(18) = 17.$$

The computation of (u_1, u_2, u_3) consists in an exchange of values between P and Q, until a value $(u_1, u_2, u_3) = (2, 4, 5)$ is found by Q.

Conclusion

Our treatment seems to extend directly to the case of non necessarily well-founded formulae. We can even consider partial strategy, and prove for instance proposition 5 by a bissimulation argument.

The approach followed in this paper leads to a (may be new) proof of cut-elimination in a strictly deterministic framework. We think that it can be extended by allowing each player to play simultaneously a finite set of moves.

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