# Higher Inductive Types as cubical sets

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## Introduction

We explain how to interpret propositional truncation as an operation on cubical sets. While it is reasonably clear how to do it for "closed" types (and the idea was presented in [1]), there is a subtle issue for this intepretation over a context.

## 1 Cubical sets

Let $\mathcal{C}$ be the following category. The objects are finite sets $I, J, K, \ldots$ thought of as finite sets of symbols/names/directions. A morphism $I \to J$ is a map $I \to \mathsf{dM}(J)$, where $J$ is the free de Morgan algebra on $J$. If $f : I \to J$ and $g : J \to K$ we write $fg : I \to K$ the composition of $f$ and $g$. We write $1 : I \to I$ the identity map. A *cubical set* is a presheaf on $\mathcal{C}^{opp}$.

A cubical set $X$ is thus given by a family of sets $X(I)$ together with a restriction map

$$X(I) \to X(J)$$

$$u \longmapsto uf$$

such that $u1 = u$ and $(uf)g = u(fg)$. We think of the elements of $X(I)$ for $I = i_1, \ldots, i_n$ as element $u = u(i_1, \ldots, i_n)$ depending on $i_1, \ldots, i_n$ and the restriction operation $uf$ as a substitution. For instance an element $u = u(i, j)$ in $X(i, j)$ represents a square, and if $(i0) : i, j \to j$ is the map sending $i$ to 0, then $u(i0)$ is the face $u(0, j)$ of this square. If $(i = j) : i, j \to j$ is the map sending $i$ to $j$ then $u(i = j)$ is the diagonal $u(j, j)$.

Any topological space $X$ defines a cubical set, by taking $X(I)$ to be the set of continuous maps $[0, 1]^I \to X$.

If $f : I \to J$ and $A$ is any de Morgan algebra, then there is a map $A^J \to A^I$ since we have a canonical map $A^J \to A^{\mathsf{dM}(J)}$. In particular we can define a functor $\mathcal{C}^{opp} \to \mathsf{Top},\ I \longmapsto [0, 1]^I$.

The *interval* $\mathbf{I}$ is the cubical set defined by $\mathbf{I}(J) = \mathsf{dM}(J)$. This defines a functor since any map $I \to \mathsf{dM}(J)$ corresponds exactly to one map of de Morgan algebra $\mathsf{dM}(I) \to \mathsf{dM}(J)$. We can think of $\mathbf{I}$ as an abstract representation of the unit real interval $[0, 1]$, and we have operations $i \wedge j,\ i \vee j,\ 1 - i$ that are abstract representations of the operations $\min(i, j),\ \max(i, j),\ 1 - i$.

We write $\vdash_I A$ if $A$ is a preshaf on the slice category $\mathcal{C}^{opp}/I$. If $I$ is empty, we get back a cubical set. If $I = i$ then $A = A(i)$ represents a "line" connecting the cubical sets $A(0)$ and $A(1)$. In general, if $I = i_1, \ldots, i_n$ then $A$ represents a hypercube. Concretely, $A$ is given by a family of sets $Af$ indexed by $f : I \to J$ together with a family of restriction maps $u \longmapsto ug,\ Af \to Afg$ for $g : J \to K$ such that $u1 = u$ and $(ug)h = u(gh)$ if $h : K \to L$. If $\vdash_I A$ and $f : I \to J$ we can consider $\vdash_J Af$ which is defined by $(Af)g = A(fg)$ for $g : J \to K$.

We write $\vdash_I a : A$ to mean that $a$ is an element in the set $A1$. It then defines a family of elements $af$ in $Af$.

If $f : I \to J$ is an inclusion then this map has a retraction in $\mathcal{C}$ and hence the corresponding map $A(I) \to A(J),\ u \longmapsto uf$ is an injection for any cubical set $I$. We adopt the convention of writing simply $u$ for $uf$ in this case, if $u$ is in $A(I)$.

## 2 Composition

If we have $\vdash_I A$, we say that $A$ has *composition* if there exists a family of composition operations $\mathsf{comp}^j_{\vec{u}}(u_0)$ in $Af(j1)$ for $u_0$ in $Af(j0)$ and $u_\alpha$ in $Af\alpha$, which satisfies the uniformity condition

$$\mathsf{comp}^j_{\vec{u}}(u_0)g = \mathsf{comp}^k_{\vec{u}(g,j=k)}(u_0 g)$$

if $g : J - j \to K$ and $k$ is not in $K$. We also require the regularity condition $\mathsf{comp}^j_{\vec{u}}(u_0) = u_0$ if $Af$ and $\vec{u}$ are independent of $j$, i.e. $Af(j0)\iota_j = Af$ and $\vec{u}(j0)\iota_j = \vec{u}$.

In the case where $\vec{u}$ is empty we have a *transport* function $Af(j0) \to Af(j1)$.

## 3 Circle

We describe $\mathsf{S}^1$ as a higher inductive type.

We need to define a set $\mathsf{S}^1(I)$ for each finite set of symbols $I$. An element of this set is

1. either $\mathsf{base}$
2. or $\mathsf{loop}\ \varphi$ where $\varphi$ is an element of $\mathsf{dM}(I)$ different from $0, 1$
3. or of the form $\mathsf{comp}^i(\vec{u}, u_0)$ where $i$ not in $I$ and $u_0$ is in $\mathsf{S}^1(I)$ and $u_\alpha$ is in $\mathsf{S}^1(I_\alpha, i)$, and such that $u_\alpha(i0) = u_0\alpha$

Thus the element of $\mathsf{S}^1(I)$ are defined by these generators together with the relation that we have $\mathsf{comp}^i(\vec{u}, u_0) = u_0$ if all $u_\alpha$ are independent of $i$.

We define recursively on $u$ in $\mathsf{S}^1(I)$ at the same time the element $uf$ in $\mathsf{S}^1(J)$ if $f : I \to J$. In this way we interpret $\vdash \mathsf{S}^1$ with $\vdash \mathsf{base} : \mathsf{S}^1$ and $\vdash_i \mathsf{loop}\ i : \mathsf{S}^1$. For the cubical set $\mathsf{S}^1$ it is decidable if $u \in \mathsf{S}^1(I)$ is independent or not of some element $i$ in $I$.

Given $\mathsf{S}^1 \vdash F$ it is also possible to define a section $\vdash s : (\Pi x : \mathsf{S}^1)F(x)$ if we give $\vdash b : F\ \mathsf{base}$ and $\vdash_i l : F\ (\mathsf{loop}\ i)$. Furthermore, we have $\vdash s\ \mathsf{base} = b : F\ \mathsf{base}$ *and* $\vdash_i s\ (\mathsf{loop}\ i) = l : F\ (\mathsf{loop}\ i)$.

## 4 Propositional truncation

We describe now the propositional truncation as an element of type $U \to U$. We define $U(I)$ to be the set of small $A$ such that $\vdash_I A$. Concretely $A$ is a family of small sets $Af$ with restriction maps $Af \to Afg,\ u \longmapsto uf$ satistying $u1 = u$ and $(uf)g = u(fg)$. Given such a structure $A$, we have to define a family of sets $\mathsf{inh}(A)f$. An element of $\mathsf{inh}(A)f$ is defined inductively as before, it is

1. either $\mathsf{inc}\ u$ with $u$ in the set $Af$
2. or $\mathsf{squash}\ \varphi\ u_0\ u_1$ with $\varphi$ in $\mathsf{dM}(J)$ and $u_0,\ u_1$ in $\mathsf{inh}(A)f$
3. or of the form $\mathsf{comp}^j(\vec{u}, u_0)$ where $j$ not in $J$ and $u_0$ is in $\mathsf{inh}(A)f$ and $u_\alpha$ is in $\mathsf{inh}(A)f\iota_j\alpha$ where $\iota_j$ is the injection $J \to J, j$

It is then possible to define $ug$ in $\mathsf{inh}(A)fg$ for $g : J \to K$ by induction on $u$ in $\mathsf{inh}(A)f$. We also have $\mathsf{inh}(Af)g = \mathsf{inh}(A)fg$ and hence we have defined a natural transformation $\mathsf{inh} : U \to U$.

What is subtle is the third clause, since we do not require directly $\mathsf{inh}(A)$ to have a composition operation.

Instead we have to show that the three conditions imply that $\mathsf{inh}(A)$ has composition operation if $A$ has composition operation. The issue here seems to be closely connected to Lemmas 6.2.3 and 6.2.4 in [2].

We first show that $\mathsf{inh}(A)$ has *transport*, that is, we can define $\mathsf{comp}^j(v_0)$ in $\mathsf{inh}(A)f(j1)$ if $v_0$ is in $\mathsf{inh}(A)f(j0)$. Let $tr$ be the transport function $v \longmapsto \mathsf{comp}^j(v_0)$. The definition of $tr(v_0)$ is done by induction on $v_0$:

1. in the case where $v_0$ is in the form $\mathsf{inc}(a_0)$ we only need that $A$ has transport

2. in the case where $v_0$ is of the form $\mathsf{squash}\ \varphi\ u_0\ u_1$ then $tr(v_0)$ is $\mathsf{squash}\ \varphi\ tr(u_0)\ tr(u_1)$

3. in the case of $v_0$ is of the form $\mathsf{comp}^j(\vec{u}, u_0)$ where $j$ not in $J$ then $tr(v_0)$ is $\mathsf{comp}^j(tr(\vec{u}), tr(u_0))$

In the general case where we have to define $\mathsf{comp}^j_{\vec{u}}(u_0)$, we take $v_0 = \mathsf{comp}^j(u_0)$ in $\mathsf{inh}(A)f(j1)$ and $v_\alpha = \mathsf{comp}^k(u_\alpha)$ for $\mathsf{inh}(A)f\alpha(j = j \vee k)$ with $k$ not in $J$. We define then

$$\mathsf{comp}^j_{\vec{u}}(u_0) = \mathsf{comp}^j(\vec{v}, v_0)$$

which is in $\mathsf{inh}(A)f(j1)$.

The definition of the suspension operation is similar. The definition of the push-out operation involves new complications for defining the transport function.

# References


[1] M.Bezem, Th. Coquand and S. Huber. A model of type theory in cubical sets. Types proceeding, 2013.

[2] B. van der Berg and R. Garner. Topological and simplicial models of identity types. ACM Transactions on Computational Logic (TOCL), Volume 13, Number 1 (2012).