Home Work I

Basic exercices.

1. Let E be a set with 8 elements. How many binary relations are there on E? How many reflexive relations are there? How many symmetric relations are there? How many reflexive symmetric relations are there? (4p)

2. Let $f : N \to N$ be a one-to-one function. Show that for all k, there exists M such that for all $n \geq M$ we have $f(n) \geq k$. (2p)

3. Let E be a set and R, S two binary relations on E. If R and S are transitive, is $R \cup S$ transitive? (2p)

4. We consider the following grammar G: the set N is $\{S\}$ the set T is $\{a\}$ and the productions are

   $$S \to SS \mid a$$

   What is $L(G)$? (2p)

More difficult.

1. A simple representation a non deterministic computation is a *transition system* which is a set A with a binary relation written usually $\to \subseteq A^2$. Intuitively the elements of A are possible states of a computation and $p {\to} p'$ means that the computation may proceed from state p to state p'. We say that another relation S on A is a *bisimulation* iff we have

   - if $p\ S\ q$ and $p {\to} p'$ then there exists q' such that $q {\to} q'$ and $p'\ S\ q'$,
   - if $p\ S\ q$ and $q {\to} q'$ then there exists p' such that $p {\to} p'$ and $p'\ S\ q'$.

   (Intuitively, if $p\ S\ q$ then whatever p can do, q can do it also and whatever q can do, p can do it also.)

   We define now a relation $p \sim q$ by "there exists a bissimulation S such that $p\ S\ q$."

   Show that the relation $\sim$ is itself a bisimulation.

   If $A = \{a, b, c, d\}$ and $a {\to} a$, $b {\to} b$, $c {\to} c$, $c {\to} d$. Show that $a \sim b$ and not $b \sim c$. (Intuitively, c may go to a state where there is a "deadlock", but not b.) (6p)

2. Let U be a set of cardinal n. We define for $A \subseteq U$

   $$\mu(A) = |A|/n$$

   Show that we have $0 \leq \mu(A) \leq 1$ for all A and that $\mu(A) > 0$ if $A \neq \emptyset$. Show that

   $$\mu(A \cup B) \leq \mu(A) + \mu(B).$$

   We define $d(A, B) = \mu(A \oplus B)$ with $A \oplus B = A - B \cup B - A$. Show that d is a *distance* that is that we have $d(A, B) = 0$ iff $A = B$ and $d(A, B) = d(B, A)$ and

   $$d(A, C) \leq d(A, B) + d(B, C)$$

   for all $A, B, C \subseteq U$. (4p)