Replacement in models of univalence

Introduction

The following replacement principle can be formulated in homotopy type theory. If $f : A \to B$ and A is small, and B is locally small, i.e. each identity type of B is equivalent to a small type, then the image of A is small. So if we have $A : \mathcal{U}$ and $R : B \to B \to \mathcal{U}$ with $\Sigma_{y:B} R\ b\ y$ contractible for all $b : B$, then we can find $X : \mathcal{U}$ and $\mathsf{inc} : A \to X$ and $g : X \to B$ with $g \circ \mathsf{inc} = f$ and g embedding, i.e. the maps $x_0 = x \to gx_0 = gx$ are embedding.

This principle is important and the reference [1] contains several applications. It is due to E. Rijke [3]; the proof is non trivial and uses the join construction.

The goal of this note is to look at what this principle means in models of univalence, and to justify this principle directly in the model.

1 Justification of the replacement principle

We build the image X by an inductive process and at the same time the map $g : X \to B$.

An element of $X\rho$ for ρ in $\Gamma(J)$, is

1. either an element $\mathsf{inc}\ a$, with a in $A\rho$ and then $g(\mathsf{inc}\ a) = f\ a$
2. or $\mathsf{ext}(x_0, \psi, x, \omega)$ with x_0 in $X\rho$ and $\psi \neq 1$ in $\Phi(J)$ and x of extent ψ and ω a partial element of extent ψ in $R\rho\ (gx_0)\ (gx)$.

We define then

$$g\ \mathsf{ext}(x_0, \psi, x, \omega) = \mathsf{ext}(gx_0, \psi, gx, \omega)$$

If $\alpha : K \to J$ is a restriction map, we define $\mathsf{ext}(x_0, \psi, x, \omega)\alpha = \mathsf{ext}(x_0\alpha, \psi\alpha, x\alpha, \omega\alpha)$ if $\psi\alpha \neq 1$ and $\mathsf{ext}(x_0, \psi, x, \omega)\alpha = x\alpha$ if $\psi\alpha = 1$.

This defines X and the map $g : X \to B$ and $\mathsf{inc} : A \to X$ and we have $g \circ \mathsf{inc} = f$ strictly.

By construction, we have $\Sigma_{x:X} gx_0 = gx$ is contractible and hence, if X is fibrant, each maps $x_0 = x \ \to\ gx_0 = gx$ are equivalence.

One main result of this note is the following observation. We use the notion of homogeneous composition and of transport operation from [2].

Theorem 1.1. *If B has a (homogeneous) composition operation then X has a (homogeneous) composition operation.*

Proof. The idea is that if we have x_0 and a partial path $\omega : x \to x_0$ then this gives a path $g\omega : gx \to gx_0$ in B which gives a proof in $R\rho\ (gx)\ (gx_0)$ and then we can use the extension opeartion to show that x can be extended to a total element. □

Lemma 1.1.1. *If A has a transport operation, then for any $\gamma : \Gamma^{\mathbb{I}}$ constant on ψ and any a_0 in $A\gamma(0)$ we can find $a(i)$ in $A\gamma(i)$ such that $a(0) = a_0$ and $a(0) = a(i)$ on ψ.*

Proof. We define $\gamma_i(j) = \gamma(i \wedge j)$ which is constant on $\psi \vee (i = 0)$ and with a_0 in $A\gamma_i(0)$. By transport we find $a(i)$ in $A\gamma_i(1) = A\gamma(i)$ such that $a(i) = a_0$ on $\psi \vee (i = 0)$. □

Theorem 1.2. *If A has a transport operation and B is fibrant over Γ then X has a transport operation.*

Proof. We take γ in $\Gamma^{\mathbb{I}}$ which is constant on ψ, and we define the transport of an element in $X\gamma(0)$. This is by induction on this element.

If it is $\mathsf{inc}\ a$ then we transport a.

If it is $\mathsf{ext}(x_0, \varphi, z_0, \omega_1)$ then we transport x_0 and z_0 by induction. We then have a dependent path $x(i)$ in $X\gamma(i)$ and $z(i)$ in $X\gamma(i)$ of extent φ. This induces dependent paths $gz(i)$ on ψ and $gx(i)$ and in this way, since R is fibrant, we transport $\omega_0 : R\gamma(0)\ (gz_0)\ (gx_0)$ to $\omega_1 : R\gamma(1)\ (gz_1)\ (gx_1)$. □

Corollary 1.2.1. *If A has a transport operation and B is fibrant over Γ then X is fibrant over Γ.*

2 Particular case: propositional truncation

Elements are $x, \psi_1, x_1, \ldots, \psi_n, x_n$ and this is equal to $x_;, \ldots, \psi_n, x_n$ if ϕ_l is equal to 1

We do not have a map $\|A\| \to A$

$\|A_0\| \times \|A_1\| \to \|A_0 \times A_1\|$

References

[1] Symmetry. `https://unimath.github.io/SymmetryBook/`. Accessed: AccessDate.

[2] Th. Coquand, Simon Huber, and Anders Mörtberg. On higher inductive types in cubical type theory. In *LICS '18 Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 255–264, 2018.

[3] Egbert Rijke. The join construction, 2017.