A coherent theory of strict Henselian ring

The goal of this note is to present a small variation of a result of Gavin Wraith. We give a coherent theory for strict Henselian ring.

For the formulation of this theory, we need the notion of universal decomposition algebra $L = A[x_1, \ldots, x_n] = A[X_1, \ldots, X_n]/I$ of a monic polynomial $f = X^n - a_1X^{n-1} + a_2X^{n-2} - \ldots$ in $A[X]$ where I is the ideal generated by

$$a_1 = s_1, \ \ldots, \ a_n = s_n$$

where $s_1, \ldots, s_n$ in $A[X_1, \ldots, X_n]$ are the elementary symmetric polynomials. It is well known that $L = A[x_1, \ldots, x_n]$ is freely generated as a module over A by the Cauchy modules and it follows from this that L is a faithfully flat extension of A.

We say that f is *unramifiable* iff $1 = (f'(x_1), \ldots, f'(x_n))$ in L. Since

$$D(f'(x_1), \ldots, f'(x_n)) = \Delta(f) = D(d_1(f), \ldots, d_n(f))$$

where $d_i(f) = s_i(f'(x_1), \ldots, f'(x_n))$ it follows from L faithfully flat that this condition is equivalent to $\Delta(f) = 1$ *in* A

Lemma 0.0.1. *If* $f = (X - a_1) \ldots (X - a_p)g$ *in* $A[X]$ *and* $\Delta(f) = 1$ *and* A *is local, then* $f'(a_1)$ *is invertible or* $\ldots$ *or* $f'(a_p)$ *is invertible or* $\Delta(g) = 1$.

Proof. We write $h = (X - a_1) \ldots (X - a_p)$.

We have in $A[x_1, \ldots, x_n]$ decomposition algebra of g

$$1 = D(f'(a_1), \ldots, f'(a_p), h(x_1)g'(x_1), \ldots, h(x_n)g'(x_n))$$

and hence

$$1 = D(f'(a_1), \ldots, f'(a_p), g'(x_1), \ldots, g'(x_n))$$

and from this follows

$$1 = D(f'(a_1), \ldots, f'(a_p)) \vee \Delta(g)$$

in $A[x_1, \ldots, x_n]$. Since the decomposition algebra is faithfully flat, we get

$$1 = D(f'(a_1), \ldots, f'(a_p)) \vee \Delta(g)$$

in A and since A is local we have the conclusion. □

Lemma 0.0.2. *If A is local and any monic unramifiable polynomial has a root in A then any monic unramifiable polynomial has a simple root in A.*

Proof. Assume that A is local and any monic unramifiable polynomial has a root in A. Let f be a monic unramifiable polynomial. It has a root a_1 and we can write $f = (X - a_1)f_1$. By the Lemma, a_1 is a simple root of f or $\Delta(f_1) = 1$. If $\Delta(f_1) = 1$ then it has a root a_2 and $f = (X - a_1)(X - a_2)f_2$. Then by the Lemma again, a_1 or a_2 is a simple root of f of $\Delta(f_2) = 1$, and so on until we find a simple root of f. □

The coherent theory is then that AA is local and that any monic unramifiable polynomial has a root
By the second Lemma, we can ask instead that any monic unramifiable polynomial has a simple root.

Using the first Lemma, one can also show that if to be invertible is decidable then the discrete residual field is separably closed and the ring is Henselian.