Types as Kan Simplicial Sets

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Types as Homotopy Types

1980s various models of dependent types as domains

1993 Hofmann-Streicher model of types as groupoids (at first motivated by an independence result in type theory)

2006 S. Awodey, M. Warren Quillen model structures and equality types

2006 Streicher types as Kan simplicial sets

Types as Homotopy Types

2006 Voevodsky "A very short note on homotopy lambda-calculus"

2006 Voevodsky "Foundation of Mathematics and Homotopy Theory", talk at IAS

2009 Voevodsky model of the univalence axiom, stratification of types with homotopy levels and how to organize mathematical libraries in type theory

Types as Homotopy Types

Given the groupoid model, it was quite natural to try to model types as weak ω-groupoids

However even the definition of weak ω-groupoid is complicated (coherence conditions)

Instead it is much simpler to interpret types as Kan simplicial sets (and this is a natural refinement of the setoid interpretation)

However as we shall see, the fact that this forms a model of type theory uses in an essential way classical logic (even before checking the univalence axiom)

Types as Homotopy Types

Goals of this talk

(0) the relation bewteen simplical sets model and logical relations

(1) to explain the constructivity issues with the Kan simplicial set model

(2) to provide a model of types as Kan simplical set in a *constructive framework*

We analyze only the case of dimension $\leqslant 1$ but in a way which hopefully generalizes to higher dimensions

Set-Theoretic versus Combinatorial Topology

Forgotten distinction; quite important however in 1910-20

Combinatorial topology, constructive, versus

Set theoretic, non constructive

Representation of a space as a *set* of points, often uncountable

Brouwer was one of main investigator, even considered as the founder, of combinatorial topology

Cf. J.P. Dubucs *L.E.J. Brouwer: topologie et constructivisme*

Simplicial Maps and Logical Relations

Surface represented by

(0) a (finite) set of points X

(1) a (finite) set of lines $X^{(1)}$ and

(2) a (finite) set of triangles $X^{(2)}$

In Set Theory, this becomes an uncountable set

Simplicial Maps and Logical Relations

Each line α has a source $d_0\ \alpha$ and a target $d_1\ \alpha$

Each triangle θ has three faces $d_0\ \theta,\ d_1\ \theta,\ d_2\ \theta$

Definitions get simpler by allowing *degenerated* lines $\eta_0\ a$ and degenerated triangles $\eta_0\ \alpha,\ \eta_1\ \alpha$

Simplicial Maps and Logical Relations

If we have two surfaces $X, X^{(1)}, X^{(2)}$ and $Y, Y^{(1)}, Y^{(2)}$ one can represent in a purely combinatorial way a continuous function as a map $f : X \rightarrow Y$ *together with* $f^{(1)} : X^{(1)} \rightarrow Y^{(1)}$ and $f^{(2)} : X^{(2)} \rightarrow Y^{(2)}$ such that

$$f^{(1)}\ (\eta_0\ a) = \eta_0\ (f\ a), \qquad d_i\ (f^{(1)}\ \alpha) = f\ (d_i\ \alpha)$$

$$f^{(2)}\ (\eta_i\ \alpha) = \eta_i\ (f^{(1)}\ \alpha), \qquad d_i\ (f^{(2)}\ \theta) = f^{(1)}\ (d_i\ \theta)$$

Simplicial Maps and Logical Relations

If we look only at the dimension $\leqslant 1$ and we consider $X^{(1)}$ (resp. $Y^{(1)}$) as defining a relation R on X (resp. S on Y) we see that this is a refinement of the notion of functions $f : X \to Y$ *preserving* relations

$$\forall x_0\ x_1 : X \qquad R\ x_0\ x_1 \to S\ (f\ x_0)\ (f\ x_1)$$

The refinement is in asking $f^{(1)}\ (\eta_0\ a) = \eta_0\ (f\ a)$

Thus we can think the model of types as simplicial sets as a refinement and extension of the logical relation model of type theory

Simplicial Maps and Logical Relations

If R is reflexive and S is an equivalence relation then the relation $T\ f_0\ f_1$

$$\forall x_0\ x_1 : X \qquad R\ x_0\ x_1 \to S\ (f_0\ x_0)\ (f_1\ x_1)$$

is an equivalence relation

This corresponds to the result that Y^X satisfies the Kan property whenever Y does

Simplicial Maps and Logical Relations

There are two natural distinct proofs of transitivity: give $T\ f\ g$ and $T\ g\ h$ and $R\ a\ c$ for proving $S\ (f\ a)\ (h\ c)$ we can

1. either use $S\ (f\ a)\ (g\ a)$ and $S\ (g\ a)\ (h\ c)$

2. or use $S\ (f\ a)\ (g\ c)$ and $S\ (g\ c)\ (h\ c)$

to conclude $S\ (f\ a)\ (h\ c)$

Which proofs should we choose? We present a possible analysis of this question later

Simplicial sets

In this talk we concentrate on dimension $\leqslant 1$

The Kan condition in this case corresponds to *symmetry* and *transitivity*

The model we describe can be seen as a refinement of the *setoid* model of type theory

Hopefully the same structure extends to all dimensions

Types as Simplicial sets

$\Gamma \vdash$ $\qquad\qquad$ $\Gamma \vdash A$ $\qquad\qquad$ $\Gamma \vdash a : A$

A context should be interpreted as a Kan simplicial set

A type $\Gamma \vdash A$ should be interpreted as a (Kan) fibration

An element $\Gamma \vdash a : A$ should be interpreted as a section of this fibration

Types as Simplicial sets

If $\sigma : \Gamma$ then $A\sigma$ is a set

If we have a line $\alpha : \sigma_0 \rightarrow \sigma_1$ then $A\alpha$ is a set of lines γ such that $d_i\ \gamma$ is in $A\sigma_i$

If $\Gamma \vdash a : A$ then $a\sigma$ is an element of $A\sigma$ and $a\alpha$ is a path in $A\alpha$

Furthermore we have the degeneracy conditions: if $\alpha = \eta_0\ \sigma$

$A(\eta_0\ \sigma)$ is the set of lines of $A\sigma$

$a(\eta_0\ \sigma) = \eta_0\ (a\sigma)$

Types as Simplicial sets

If $\Gamma \vdash A$ and $\Gamma.A \vdash B$ and $\sigma : \Gamma$ then $(\Pi\ A\ B)\sigma$ is the set of functions f with a function $f^{(1)} = \eta_0\ f$ such that

(1) if $u : A\sigma$ then $f\ u : B(\sigma, u)$

(2) if $\gamma : u_0 \to u_1$ in $A(\eta_0\ \sigma)$ then $f^{(1)}\ \gamma : f\ u_0 \to f\ u_1$

(3) we have $f^{(1)}\ (\eta_0\ u) = \eta_0\ (f\ u)$

Types as Simplicial sets

If $\Gamma \vdash A$ and $\Gamma.A \vdash B$ and $\alpha : \sigma_0 \rightarrow \sigma_1$ then $(\Pi\ A\ B)\alpha$ is the set of triple f_0, f_1, λ with $f_i : (\Pi\ A\ B)\sigma_i$ such that

if $\omega : u_0 \rightarrow u_1$ is in $A\alpha$ then $\lambda\ \omega$ is a path $f_0\ u_0 \rightarrow f_1\ u_1$ in $B(\alpha, \omega)$

$d_i\ (f_0, f_1, \lambda) = f_i$

The simplicial set $\Gamma.A$ has for points the pairs σ, u with $\sigma : \Gamma$ and $u : A\sigma$ and for lines the pairs α, ω with $\alpha : \sigma_0 \rightarrow \sigma_1$ and $\omega : A\alpha$

This defines a model where types are simplicial sets. We have to check that the Kan property is preserved by the type forming operation.

Types as Simplicial sets

We have

$(\lambda t)\sigma\ u = t(\sigma, u)$

$(\lambda t)\alpha\ \omega = t(\alpha, \omega)$

$\eta_0\ (\sigma, u) = (\eta_0\ \sigma, \eta_0\ u)$

Types as Kan Simplicial sets

This is a direct generalization of R. Gandy 1956 interpretation of extensional type theory in intensional type theory

Each type is interpreted by a set with a relation

One has to check by induction on the types that this relation is an equivalence relation

An earlier simpler instance of this interpretation

B. Russell *The Theory of Implications* 1906, American Journal of Mathematics

Types as Kan Simplicial sets

Problem: the fact that this forms a model of type theory uses *classical logic* in an essential way

Most definitions are by cases on whether a given simplex is degenerated or not

But the condition of being degenerated is not decidable in general

Even the result: Y^X Kan if Y Kan

seems to require classical logic in an essential way

The use of classical logic is best seen for the definition of Kan fibration

Kan fibration

Let $p : F \to B$ be a Kan fibration

Over any point $b : B$ we have the fiber $F(b) = p^{-1}(b)$ a simplical set

If we have a path $\alpha : b_0 \to b_1$ in B the Kan filling condition gives a *set-theoretic* map $F(\alpha)^+ : F(b_0) \to F(b_1)$

Classically using the other Kan filling conditions there is no problem to extend this map to a map of *simplicial sets*

Similarly we have $F(\alpha)^- : F(b_1) \to F(b_0)$ and the pair $F(\alpha)^+,\ F(\alpha)^-$ is an homotopy equivalence which shows that $F(b_0)$ and $F(b_1)$ have the same homotopy type

Kan fibration

Constructively, it does not seem possible to derive all these properties from the usual definition of Kan fibration (even giving the filling explicitely)

Instead we have to incorporate the important properties of Kan fibrations in the definition

All the problems are in checking the degeneracy conditions

Types as Kan Simplicial sets

To summarize:

Classical logic is used in an essential way in checking that Kan simplicial sets form a model of type theory (even before checking the univalence axiom)

This is a quite interesting use of classical logic

However we don't expect classical logic to be essential

Furthermore this model should generalize/refine the setoid model

We are presenting a constructive version of this model

Kan fibration

The constructive condition can be expressed concisely as follows using A. Joyal's notion of *left fibration*

Consider the pull-back $B^I \times_B^0 F$ of $F \to B$ and $B^{d_0} : B^I \to B$

We have a map $\langle d_0, p \rangle : F^I \to B^I \times_B^0 F$

We require that this map has a section s

Furthermore there are two constant maps $c_0 : F \to F^I$ and $c_1 : F \to B^I \times_B^0 F$ and we ask that $c_0 = s \circ c_1$

Kan fibration

We require 4 compositions operation (and not only 3)

$\mathsf{comp}_0\ \alpha\ \beta : a_1 \rightarrow a_2$ for $\alpha : a_0 \rightarrow a_1,\ \beta : a_0 \rightarrow a_2$

$\mathsf{comp}_1\ \alpha\ \beta,\ \ \mathsf{comp}'_1\ \alpha\ \beta : a_0 \rightarrow a_2$ for $\alpha : a_0 \rightarrow a_1,\ \beta : a_1 \rightarrow a_2$

$\mathsf{comp}_2\ \alpha\ \beta : a_0 \rightarrow a_1$ for $\alpha : a_0 \rightarrow a_2,\ \beta : a_1 \rightarrow a_2$

Equations

$\mathsf{comp}_0\ (\eta_0\ a)\ \beta = \beta \quad \mathsf{comp}_1\ (\eta_0\ a)\ \beta = \beta$

$\mathsf{comp}'_1\ \alpha\ (\eta_0\ b) = \alpha \quad \mathsf{comp}_2\ \alpha\ (\eta_0\ b) = \alpha$

Kan fibration

(Heuristic comment)

We know that these equations should hold in the model

For instance we have a proof p of

$\mathsf{Id}\ a_0\ a_1 \to \mathsf{Id}\ a_0\ a_2 \to \mathsf{Id}\ a_1\ a_2$ of the form

$\mathsf{Id}\ a_0\ a_1 \to C(a_0) \to C(a_1)$

and $p\ \alpha\ \beta$ is convertible to β if α is reflexivity

So *if* we have a model this model should satisfies these equations

Kan fibration

To introduce two composition operations

$\mathsf{comp}_1\ \alpha\ \beta,\quad \mathsf{comp}'_1\ \alpha\ \beta : a_0 \to a_2$ for $\alpha : a_0 \to a_1,\ \beta : a_1 \to a_2$

"solves" the problem of the non canonical definition we have to make in checking the transitivity condition in Gandy's model

We have actually *two* different notions of composition which correspond to the two different proofs of transitivity for function spaces

Kan fibration

We require $A\alpha^+ : A\sigma_0 \to A\sigma_1$ with $A\alpha \uparrow u : u \to A\alpha^+ \ u$ for any $u : A\sigma_0$

Continuity condition: given any commuting "square" $\gamma_i : \sigma_i \to \delta_i$ between $\alpha : \sigma_0 \to \sigma_1$ and $\beta : \delta_0 \to \delta_1$ then any line $\omega : u \to v$ in $A\gamma_0$ extends to a square $u \to A\alpha^+ u, \quad v \to A\beta^+ v$

Furthermore if the given square is degenerate and $\omega = \eta_0 \ u$ then the resulting square is also degenerate

Also if $\alpha = \eta_0 \ \sigma$ and $\beta = \eta_0 \ \delta$ then the resulting square is degenerate

In particular $A(\eta_0 \ \sigma) \uparrow u = \eta_0 \ u$

Kan fibration

The map $A\alpha^{+} : A\sigma_0 \rightarrow A\sigma_1$ can be seen as a *coercion* between the types $A\sigma_0$ and $A\sigma_1$

The line $A\alpha \uparrow u : u \rightarrow A\alpha^{+}u$ expresses a kind of *coherence* condition

The set $A\alpha$ should be thought of as the type of heterogeneous equalities between elements in $A\sigma_0$ and $A\sigma_1$

Kan fibration

The crucial computations are for the product types

To simplify the notation (and in preparation of the interpretation of the univalence axiom) one introduces an universe of small Kan simplifical sets.

We should have if $\alpha : X_0 \rightarrow X_1$ the following filling maps

$\alpha^+ : X_0 \rightarrow X_1 \qquad \alpha^- : X_1 \rightarrow X_0$

$\alpha \uparrow a_0 : a_0 \rightarrow \alpha^+ a_0 \qquad \alpha \downarrow a_1 : \alpha^- a_1 \rightarrow a_1$

$(\eta_0\ X)^+\ u = u \qquad (\eta_0\ X)^-\ v = v$

$(\eta_0\ X) \uparrow u = \eta_0\ u \qquad (\eta_0\ X) \downarrow v = \eta_0\ v$

Kan fibration

Definition of $(\Pi\ \alpha\ \beta)^+$ and $(\Pi\ \alpha\ \beta) \uparrow$

$$(\Pi\ \alpha\ \beta)^+\ f\ v = \beta\ (\alpha \downarrow v)^+(f\ (\alpha^-\ v))$$

For $(\Pi\ \alpha\ \beta) \uparrow$ we need the Kan condition on triangles. For $\omega : a \to b$ we define

$$\delta = \mathsf{comp}_2\ \omega\ (\alpha \downarrow b) \text{ such that } \delta : a \to \alpha^-\ b$$

$$(\Pi\ \alpha\ \beta) \uparrow f\ \omega = \mathsf{comp}_1'\ (\eta_0\ f\ \delta)\ (\beta\ (\alpha \downarrow b) \uparrow f(\alpha^- b))$$

Kan fibration

The degeneracy conditions are used to prove that

$$g = (\Pi\ \alpha\ \beta)^{+}\ f$$

is continuous: we have to define $\eta_0\ g\ \omega : g\ v_0 \rightarrow g\ v_1$ given $\omega : v_0 \rightarrow v_1$ in such a way that

$$\eta_0\ g\ (\eta_0\ v) = \eta_0\ (g\ v)$$

This is only possible by the conditions required on Kan fibrations

Organizing the model

We have a graded combinatory algebra of values that can be points or lines

The operators d_i and η_0 are combinatory algebra morphisms

$d_0\ (\lambda\ \omega) = d_0\ \lambda\ (d_0\ \omega) \qquad d_1\ (\lambda\ \omega) = d_1\ \lambda\ (d_0\ \omega) \qquad \eta_0\ (f\ a) = \eta_0\ f\ (\eta_0\ a)$

We have also

$d_i\ (\Pi\ \alpha\ \beta) = \Pi\ (d_i\ \alpha)\ (d_i\ \beta) \qquad \eta_0\ (\Pi\ u\ f) = \Pi\ (\eta_0\ u)\ (\eta_0\ f)$

Organizing the model

If $u : X$ then $\eta_0\ u : \eta_0\ X$ and if $\omega : \alpha$ then $d_i\ \omega : d_i\ \alpha$

$(\lambda t)\sigma\ u = t(\sigma, u)$

$(\lambda t)\alpha\ \omega = t(\alpha, \omega)$

$\eta_0\ (\sigma, u) = (\eta_0\ \sigma, \eta_0\ u)$

Exploring the model

We get in this way a concrete picture of the Kan simplicial set model

Example: interpretation of C. Paulin's elimination rule for identity type

If we have a simplical set $X, X^{(1)}, \dots$ and $a : X$ we define a new simplical set $S, S^{(1)}, \dots$

S is the set of pairs x, α with $\alpha : a \to x$

$S^{(1)}$ is the set of triangles a, x_0, x_1

Exploring the model

The degenerate triangle gives a line $\theta : (a, \eta_0\ a) \to (x, \alpha)$

So given any dependent type $C : S \to U$ we should have

$\eta_0\ C\ \theta : C(a, \eta_0\ a) \to C(x, \alpha)$

and so $(\eta_0\ C\ \theta)^+$ is the interpretation of the elimination rule

If $C = \lambda x.N$ we get $\eta_0\ C\ \theta = \eta_0\ N$ and we get the identity function

This answers one canonicity problem for higher-order inductive types

Exploring the model

In the model we have an interval type I with $0 \quad 1 : I$ and a primitive line $\alpha : 0 \to 1$

If we have another simplicial set $X, X^{(1)}, \ldots$ with $a_0 \; a_1 : X$ and $\omega : a_0 \to a_1$ then there exists an unique $f : I \to X$ with $f \; i = a_i$ and $\eta_0 \; f \; \alpha = \omega$

Exploring the model

In the model, we also have a type X for the circle S^1

We have only one point in X

$X^{(1)}$ is $\mathbb{Z}$

The triangles are triples n_1, n_2, n with $n_1 + n = n_2$

All higher simplexes are trivial

Model of type theory

We have checked that we get in this way a model of type theory *without* using classical logic

Should suggest a way to implement a type-checker for dependent type theory with the univalence axiom

Should suggest a way to add quotient types