Sheaf Cohomology and Univalent Type Theory

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Sheaf cohomology

Cohomology groups: invariants attached to mathematical structures

Ex.: Brauer group (1927), commutative group associated to *any* field

Extension groups: Schreier (1926), group acting on an abelian group

There, 2nd cohomology group (Teichmüller 1940, 3rd cohomology group)

Leray (1940s): sheaf cohomology over topological spaces

Sheaf cohomology

Grothendieck unified these notions with the concept of *site* and *topos*

E.g. the Brauer group is a 2nd cohomology group for the small étale topos over a field and Milnor's conjecture gives a description in term of generators and relations for cohomology groups of this space with coefficient in $\mathbb{Z}/2\mathbb{Z}$

This relied on the notion of *injective* resolutions

Grothendieck alluded in an alternative approach using n-stacks

We try to show that the language of dependent type theory is (surprisingly) well adapted to describe what is going on

Topos Theory

It [the category of sheaves] functions as a kind of "superstructure of measurement", called the "Category of Sheaves" (over the given space), which henceforth shall be taken to incoorporate all that is most essential about that space. This is in all respects a lawful procedure, (in terms of "mathematical common sense") because it turns out that one can "reconstitute" in all respects, the topological space by means of the associated "category of sheaves" (or "arsenal" of measuring instruments)

Consider the set formed by all sheaves over a (given) topological space or, if you like, the formidable arsenal of all the "rulers" that can be used in taking measurements on it.

Stacks

On the "non-commutative" side, we have a good foundation work with Giraud's thesis, but this is limited to a formalism of 1-stacks, lending itself to a direct geometric expression of objects of cohomology up to dimension 2 only. The question of developing a cohomological formalism not commutative in terms of n-stacks, imperiously suggested *by numerous examples, encountered serious conceptual difficulties.*

Considering the disaffection or, to put it better, the general contempt, into which have fallen the questions of foundations in a certain world, these difficulties have never been addressed before I started to look at them a little over two years ago

Grothendieck in "Récoltes et Semailles"

Sheaf models and Higher Order Logic

Compare with

"The rising sea, Foundations of Algebraic Geometry"

Ravi Vakil

Our general approach will be as follows. I will try to tell you what you need to know, and no more. (This I promise: if I use the word "topoi", you can shoot me.)

A topos is a scary name for a category of sheaves of sets on a Grothendieck topology.

Univalent Foundations

Thus a proper language for formalization of mathematics should allow one to directly build and study groupoids of various levels and structures on them. A major advantage of this point of view is that unlike ∞-categories, which can be defined in many substantially different ways the world of ∞-groupoids is determined by Grothendieck correspondence (see [3]) , which asserts that ∞-groupoids are "the same" as homotopy types.

Combining this correspondence with the previous considerations we come to the view that not only homotopy theory but the whole of mathematics *is the study of structures on homotopy types.*

Voevodsky "An experimental library of formalized mathematics based on univalent foundations"

Cohomology and Univalent Foundations

H^1 groupoid

H^2 2-groupoid

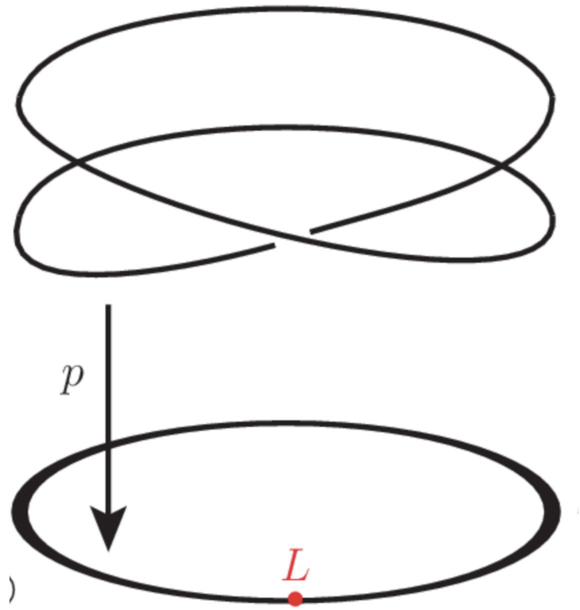
Sheaf models and Higher Order Logic

A sheaf F on a topological space is a *presheaf*, i.e. a collection of sets $F(U)$ with restriction operation $F(U) \to F(V)$ for $V \subseteq U$, satisfying a *patching* condition

If we have an open covering U_i of U and local elements a_i in $F(U_i)$ then we can patch these elements to a global element *provided* these elements are compatible, i.e. a_i and a_j coincide when restricted to $U_i \cap U_j$

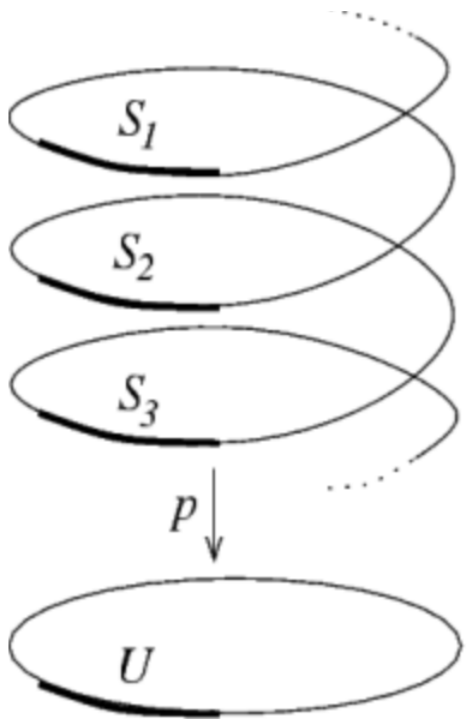
E.g. $F(U)$ is the set of continuous sections of a projection map $p : E \to B$

Möbius strip



Only *local* and no *global* sections over the circle

Helix



Only *local* and no *global* sections over the circle

Sheaf models

Grothendieck realized that the collection of “all” sheaves over a space has strong similarity with the collection of “all” sets

We can define the product $A \times B$ of two sheaves, or the function space B^A

We have a special sheaf Ω which plays the role of space of truth values

(A truth value on a space X is defined by an open inclusion $U \rightarrow X$)

In general, the law of *Excluded Middle* may not be valid!

The *Axiom of Choice* may not be valid as well

Sheaf models

All this was developped in the 60s

Kripke was then, independently, designing a new semantics for intuitionistic logic, inspired from previous work on a semantics of modal logic

(Already Beth in the 50s had developped a related semantics)

And Cohen was inventing the notion of *forcing* to show the independence of the continuum hypothesis

Sheaf models and Higher Order Logic

Dana Scott, reformulating the notion of forcing using *Boolean valued model*, realized the connection between forcing not only with Kripke models, but also with the work of Grothendieck's school

Unexpected connection between *algebraic geometry* and *intuitionism*

He also realized that the formalism of *higher logic*/Church's simple type theory is remarkably well adapted to describe what is going on in sheaf models

In particular, though the Axiom of Choice may not be valid, the Axiom of "Unique" Choice is always satisfied

Sheaf models and Higher Order Logic

A Formulation of the Simple Theory of Types, JSL, 1940

Church's system had 8 basic axioms and then

Axiom 9 "unique" choice $\psi(x) \wedge (\forall_y\ \psi(y) \rightarrow y = x) \rightarrow \psi(\iota(\psi))$

Can be rewritten as $(\exists !_x \psi(x)) \rightarrow \psi(\iota(\psi))$

Axiom 10 function extensionality

Axiom 11 global choice $\psi(x) \rightarrow \psi(\epsilon(\psi))$

Can be rewritten as $(\exists_x \psi(x)) \rightarrow \psi(\epsilon(\psi))$

Sheaf models and Higher Order Logic

The global choice operation ϵ comes from Hilbert 1922

The definite description operation ι comes from Peano and Russell provides later an analysis in *On Denoting*, MIND, 1905

This analysis is still fundamental in set theory, where a *function* is defined as a *functional relation*

Dependent Type Theory provides a *new* analysis of this operation!

Sheaf models and Higher Order Logic

In a sheaf model $\exists_{x:F}\psi(x)$ means that we have *locally* a_i in $F(U_i)$ satisfying $\psi(a_i)$

In general the elements a_i may not be *compatible* and it may not be possible to patch them together to a *global* element satisfying ψ

But this is possible if we have *unique* existence: the elements a_i should in this case be compatible

So *unique* choice is always valid in sheaf models, while *global* choice may fail

Sheaf models

We can define $\|F\|$ propositional *truncation*, image of $F \to 1$

This is a proposition/subsingleton expressing that F is inhabited

We can have $\|F\|$ without having any global element of F!

For the helix H we have $1 = \|H\|$ but there is no global element $1 \to H$

This operation $\|F\|$ is a monad, we have $F \to \|F\|$

If G is a subsingleton then we have $\|F\| \to G$ if we have $F \to G$

New notions: Torsor

G-torsor: "twisted" notion of a given group G

A sheaf T with a G-action which is *locally* isomorphic to G

But it may fail to have a global element

The helix defines a non trivial $\mathbb{Z}$-torsor

The Möbius strip defines a non trivial $\mathbb{Z}/2\mathbb{Z}$-torsor

We have $\|T\|$ but we may not have any global inhabitant

If T has a global inhabitant, it is isomorphic to G

Sheaf models

Recall that Grothendieck intended to use the collection of all sheaves of sets as a way to "measure" a space

How to distinguish the unit interval and the circle?

For the unit interval all $\mathbb{Z}/2\mathbb{Z}$-torsors are trivial

But for the circle we can find a non trivial $\mathbb{Z}/2\mathbb{Z}$-torsor, a "twisted" version of $\mathbb{Z}/2\mathbb{Z}$

Torsors

For a given group G in a topos, we can look at the collection of all G-torsors

This is actually one way to define the first cohomology group of a space for a given sheaf of groups G

$H^1(X, G)$ can be defined as the set of all G-torsors, up to isomorphisms

$H^1([0,1], \mathbb{Z}) = 0$ and $H^1(S_1, \mathbb{Z}) = \mathbb{Z}$

$H^1(S_1, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

There are, up to isomorphisms, only two $\mathbb{Z}/2\mathbb{Z}$-torsors over S_1

Torsors

Any map between two G-torsors is an isomorphism

Locally $T(U_i) \to T'(U_i)$ has some *unique* inverse

Hence we can patch these inverses to build a global inverse $T' \to T$

So, the collection of G-torsors forms a *groupoid* BG

There is a special objet T_G which is G itself, with the canonical G action

T_G is the "trivial" G-torsor

Torsors

The group $Aut(T_G)$ of automorphisms of T_G is exactly G

If we have a morphism $f : G \rightarrow H$ we can build a functor $BG \rightarrow BH$ sending T_G to T_H and which induces f via $Aut(T_G) \rightarrow Aut(T_H)$

If T is a G-torsor and $f : G \rightarrow H$ how to define the corresponding H-torsor $f(T)$?

Torsors

Deligne provides a two lines proof of this fact

The key idea is to define $f(T)$ *together with* a map $\alpha : T \to f(T)$ such that $\alpha(t \cdot x) = \alpha(t) \cdot f(x)$

This pair $f(T), \alpha$ is determined up to unique *isomorphism*

Also this pair exists locally

It is then possible to patch together these local data, *because* they are uniquely determined

(Technically, the patching is a little more subtle than patching for a sheaf, this is an example of patching for *stacks*)

Torsors

We follow here a paper of Deligne *Le symbole modéré*, 1991, 5.2, 5.3

Deligne presents the argument as if it is clear that the principle of unique choice holds, where unique means *unique up to unique isomorphism*

Internal language

Can we present these arguments in a purely logical way?

We are going to compare the proofs of

-any morphism of G-torsors $\alpha : T \to T'$ is an isomorphism

-any group morphism $f : G \to H$ can be lifted to a functor $BG \to BH$

Internal language

If $\alpha : T \to T'$ is a map of G-torsors we prove

$\psi \;:=\; \exists!_{\beta:T'\to T}\,(\beta\circ\alpha = \mathsf{id}_T \wedge \alpha\circ\beta = \mathsf{id}_{T'})$

We can consider the subsingleton $\{0 \mid \psi\}$

To prove ψ is the same as to provide an element $1 \to \{0 \mid \psi\}$

If we have $T \to \{0 \mid \psi\}$ then we have $\|T\| \to \{0 \mid \psi\}$

So, for proving a proposition ψ we can assume T inhabited!

Internal language

If $\alpha : T \to T'$ is a map of G-torsors we prove

$\psi \;:=\; \exists!_{\beta:T'\to T}\,(\beta \circ \alpha = \mathsf{id}_T \wedge \alpha \circ \beta = \mathsf{id}_{T'})$

The proof in the case where T (and then also T') is inhabited is trivial

(In this case we have isomorphisms $G \to T$ and $G \to T'$)

We can then apply unique choice to get the inverse of the map α

Internal language

-any map $f : G \to H$ can be lifted to a map $BG \to BH$

The argument *cannot* directly be represented as such neither in Simple Type Theory nor in Set Theory

It uses the fact that $f(T), \alpha$ is determined up to unique isomorphism

We need a new operation of definite description, which does not exist neither in Church's Simple Type Theory nor in Set Theory

But it is *definable* in *dependent type theory*!

Dependent Type Theory

The crucial points will be here that

-we have a universe $\mathcal{U}$ or types of "small" types

-if A is a type then and a_0, a_1 are elements of A then we can form a new type $a_0 =_A a_1$ that is the type of "equalities" or possible "identifications" of the elements a_0 and a_1

In general, $a_0 =_A a_1$, itself being a type, may have several different elements

Univalence captures what should be the intended notion of identification

For $\mathcal{U}$, it should be equivalence (natural generalisation of isomorphism)

Dependent Type Theory

Stratification of types

"Propositions" are identified with subsingleton

proposition $\prod_{a_0:A}\prod_{a_1:A}\ a_0 =_A a_1$

0-type or set $\prod_{a_0:A}\prod_{a_1:A}\ \mathsf{isProp}\ (a_0 =_A a_1)$

1-type of groupoid $\prod_{a_0:A}\prod_{a_1:A}\ \mathsf{isSet}\ (a_0 =_A a_1)$

Contractible type $A \times \mathsf{isProp}\ A$ which can be expressed as $\sum_{a:A}\prod_{x:A} a =_A x$

All these types are themselves *propositions*!

Dependent Type Theory

For any type A we can build an element of the type isProp (isContr A)

That is, we can show that any two elements of isContr A are equal

Quite remarkable that we can prove this fact from purely logical principles!

Dependent Type Theory

We need $\|A\|$ propositional *truncation*

An element of $\|A\|$ intuitively corresponds to local existence

isProp $\|A\|$ and we have

isProp $\psi \wedge (A \rightarrow \psi) \ \rightarrow \ (\|A\| \rightarrow \psi)$

Dependent Type Theory

We can define what is a *group* G in a given universe $\mathcal{U}$

It has to be a *set*/0-type with a binary operation satisfying the usual laws

We can form the collection of all groups (in a given universe) and *prove* that it is a 1-type/groupoid (using univalence)

Torsors in Dependent Type Theory

If G is a group we can then define the type BG of G-torsors

$$BG \;:=\; \sum_{T:\mathcal{U}} \|T\| \times \sum_{a:T\times G\to T} \psi(T,a)$$

where $\psi(T,a)$ is a proposition/type expresssing that a is a G-action such that $T \times G \to T \times T$ is a bijection

One can *prove* in type theory that BG is a *groupoid*

Torsors in Dependent Type Theory

Deligne's argument can then be expressed directly in type theory

Given $f : G \to H$ group morphism and $T : BG$ then we can look at

$$S(T) \ := \ \sum_{T':BH} \ \sum_{\alpha:T\to T'} \ \prod_{t:T} \prod_{x:G} \alpha(t \cdot x) = \alpha(t) \cdot f(x)$$

We show that this type is *contractible* following closely Deligne's argument

So this type, which looks like a type of structure, is actually a singleton type!

Torsors in Dependent Type Theory

Indeed, to be contractible is a *proposition*, so $\psi = \text{isContr}\ S(T)$ is a proposition

Hence for proving that a given type $S(T)$, where T is a G-torsor, is contractible we can as well assume that T has some inhabitant

This is because $T \rightarrow \psi$ is the same as $\|T\| \rightarrow \psi$ since ψ is a proposition

Torsors in Dependent Type Theory

We are then in the case where T is the trivial G-torsor, and in this case the fact that

$$S(T) \ := \sum_{T':BH} \sum_{\alpha:T\rightarrow T'} \prod_{t:T} \prod_{x:G} \alpha(t\cdot x) = \alpha(t)\cdot f(x)$$

is contractible, i.e. is inhabited and is a proposition, follows from *univalence*

In Deligne's argument, this corresponds to the fact that any two elements T'_1, α_1 and T'_2, α_2 are isomorphic, with a unique isomorphism

Torsors in Dependent Type Theory

So, we were able to represent an argument which uses the axiom of description where unique means now *unique up to unique isomorphism*

Torsors in Dependent Type Theory

Something similar holds for the result stating that for a ring R the groupoid of locally free modules of rank 1 (the *Picard group* of R) and the groupoid of $R^{\times}$-torsors are canonically equivalent

The proof in the Stacks project Lemma 21.6.1 uses an "explicit" construction with a quotient

Torsors in Dependent Type Theory

In type theory, this can be seen as a special case of the following result

$Mod(R)$ is the groupoid of R-modules and R^1 the free R-module of rank 1 so that $R^1 = R^1$ is $R^\times$

$T_R = \sum_{M:Mod(R)} \|M = R^1\|$ is the type of locally free modules of rank 1

Lemma: $T_R \rightarrow BR^\times \qquad M \mapsto (M = R^1)$ *is an equivalence*

Torsors in Dependent Type Theory

Lemma: *If A is a groupoid and $a : A$ then the canonical map*

$$T_a \to B(a = a) \qquad x \mapsto (x = a)$$

is an equivalence

The proof of this follows the same kind of reasoning: we have to show that for any $(a = a)$-torsor T the fiber $T = \sum_{(x,p):T_a}(x = a)$ is contractible

It is enough to show it when T of the trivial torsor $a = a$ and this is proved directly

This is because to be contractible is a *proposition* and any torsor is merely equal to the trivial torsor $a = a$.

Torsors in Dependent Type Theory

Once again, Deligne presents such an argument closer to the one of type theory in the 1977 notes

"Étale cohomology: starting points"

(first chapter of SGA 4 1/2)

Torsors in Dependent Type Theory

Type theoretically, this story has a nice formulation

The groupoids of the form BG, t_G can be characterised as exactly pointed 0-connected 1-types

A 0-connected means $\|A\| \times \prod_{a_0:A} \prod_{a_1:A} \|a_0 =_A a_1\|$

Torsors in Dependent Type Theory

Let $K_1(\mathcal{U})$ be the type of pointed 0-connected 1-types

Let Grp($\mathcal{U}$) the collection of groups, the same argument as before shows that $K_1(\mathcal{U}) \rightarrow \mathsf{Grp}(\mathcal{U})$ is fully faithful

And it is essentially surjective since $\Omega(BG, t_G)$ as a group is G

So this is an *equivalence*

The type $K_1(\mathcal{U})$ is equivalent to the type of groups in $\mathcal{U}$!

Torsors in Dependent Type Theory

Thus, we can present *group theory* by working with pointed 0-connected 1-types and pointed maps

This is the basis of the *Symmetry* book

https://unimath.github.io/SymmetryBook/book.pdf

M. Bezem, U. Buchholtz, P. Cagne, B. I. Dundas, D. R. Grayson

Connected types

More generally, the *same* argument gives a (new) proof of the following result.

Theorem: *If A is n-connected and B $(2n)$-type then for a in A and b in B*

$$((A,a) \to_\bullet (B,b)) \to (\Omega(A,a) \to_\bullet \Omega(B,b))$$

is an equivalence

(David Wärn, formalised in Agda by Louise Leclerc)

Dependent Type Theory

We only make use of the two Lemmas

Lemma 1: *If* A n*-connected and* B n*-type then the canonical map* $B \rightarrow B^A$ *is an equivalence*

Lemma 2: *If* A n*-connected with* $a : A$ *and* $P(x)$ *family of* $(n + k + 1)$*-types over* A *then all fibers of the evaluation map* $(\prod_{x:A} P(x)) \rightarrow P(a)$ *are* k*-types*

Particular case: if $k = -2$ then the fibers are contractible

Both n-connected and n-type are defined by induction on n

Application 1

Let $K_n(\mathcal{U})$ be the type of pointed $(n-1)$-connected n-types with $n > 0$

Let $\mathcal{U}^\bullet$ be the type of pointed types $\sum_{X:\mathcal{U}} X$

We have the loop space function $\Omega : \mathcal{U}^\bullet \to \mathcal{U}^\bullet$

This defines a map $\Omega : K_n(\mathcal{U}) \to K_{n-1}(\mathcal{U})$

It follows from the Theorem that this map is fully faithfull

Application 1

We can then show that

$\Omega : K_n(\mathcal{U}) \rightarrow K_{n-1}(\mathcal{U})$

is an equivalence for $n > 2$

Furthermore all these types are actually equivalent to the type $\mathsf{Ab}(\mathcal{U})$ of *abelian* groups

All these types define *univalent categories*

For abelian groups, *group* maps

For $K_n(\mathcal{U})$, *pointed* maps

Categories in Dependent Type Theory

Each type $K_m(\mathcal{U})$ is equivalent to the type $\mathsf{Ab}(\mathcal{U})$ (and all are 1-types)

Each category $K_m(\mathcal{U})$ is equivalent to the category $\mathsf{Ab}(\mathcal{U})$ of abelian groups

(All this is effective if we have an effective model of type theory)

For L abelian group we write $K(L,n)$ the corresponding object in $K_m(\mathcal{U})$

This is the nth Eilenberg-MacLane space associated to L

Application 2: Universal characterisation of $K(L,n)$

To give a map $K(L,n) \rightarrow_{\bullet} (A,a)$ for any n-type A is equivalent to give a map $\Omega(K(L,n)) \rightarrow_{\bullet} \Omega(A,a)$, i.e. $K(L,n-1) \rightarrow \Omega(A,a)$

And then $K(L,n-2) \rightarrow_{\bullet} \Omega^2(A,a)$, and so on, until we get $L \rightarrow \Omega^n(A,a)$

We thus get that to give a pointed map $K(L,n) \rightarrow_{\bullet} (A,a)$ is the same as to give a group morphism $L \rightarrow \Omega^n(A,a)$

Application 2

The n-sphere (S_n, b) has a similar universal characterisation: to give a map $(S_n, b) \rightarrow_\bullet (A, a)$ for *any* pointed n-type is the same as to give a map $\mathbb{Z} \rightarrow \Omega^n(A, a)$

$K(\mathbb{Z}, n)$ satisfies the same universal property, but for A n-type

It then follows that $K(\mathbb{Z}, n)$ is the nth truncation of S_n

Hence $\pi_k(S_n, b) = 0$ if $k < n$ and $\pi_n(S_n, b) = \mathbb{Z}$!

Comparison with other approaches

The proof in the HoTT book uses Freudenthal Suspension Theorem (the original proof by G. Brunerie and D. Licata uses the decode-encode method)

Actually, this was used to define $K(\mathbb{Z}, n)$ as the nth truncation of S_n

Short exact sequence

We thus get another presentation of the category of Abelian groups

What happens to the notion of short exact sequence?

How is it expressed in $K_n(\mathcal{U})$?

A sequence $(A,a) \to_\bullet (B,b) \to_\bullet (C,c)$ corresponds to a short exact sequence if, and only if, it is a *fibration sequence*

Short exact sequence

If $L \to M \to N$ is a short exact sequence of abelian groups we get the fibration sequence $K(L,n) \to_\bullet K(M,n) \to_\bullet K(N,n)$

We can then obtain the long fibration sequence

$$\dots K(L,n-1) \to_\bullet K(M,n-1) \to_\bullet K(N,n-1)$$

$$\to_\bullet K(L,n) \to_\bullet K(M,n) \to_\bullet K(N,n)$$

This can be done for *any* n

Sheaf models

So far, we have shown how to use the language of dependent type theory to express what should happen in a sheaf model

How to model types as stacks over a site?

All $(\infty, 1)$*-toposes have strict univalent universes*
M. Shulman

Constructive sheaf models of type theory
Th. C., F. Ruch and Ch. Sattler

Sheaf models

The second paper is developped in a *constructive* meta theory

It is done in a relatively *weak* meta theory

Proof theoretic strength: dependent type theory with hierarchy of universes

Weaker than second-order arithmetic

The notion of universes in an intuitionistic setting is surprisingly weak (P. Martin-Löf, P. Hancock, P. Aczel, E. Palmgren, E. Griffor, M. Rathjen)

How to recover cohomology groups?

If L is a sheaf of abelian group over a space X

Compute $K(L,n)$ in the sheaf model over X

Apply global section $\Gamma(K(L,n))$: this is a global n-type

$\pi_0(\Gamma(K(L,n))$ is $H^n(X,L)$!

This is the special case $X \to 1$ and can be generalised to $X \to Y$

How to recover cohomology groups

If we apply global section, and then π_0 to the sequence

$$\ldots K(L, n-1) \to_{\bullet} K(M, n-1) \to_{\bullet} K(N, n-1)$$

$$\to_{\bullet} K(L, n) \to_{\bullet} K(M, n) \to_{\bullet} K(N, n) \ldots$$

we get the long exact sequence

$$\ldots H^{n-1}(X, L) \to H^{n-1}(X, M) \to H^{n-1}(X, N)$$

$$\to H^{n}(X, L) \to H^{n}(X, M) \to H^{n}(X, N) \ldots$$

Summary

We have different presentations of the category of abelian groups (in a given universe)

The elements are pointed $(n-1)$-connected n-types

This is the method that Grothendieck alluded to in a letter to Larry Breen 1975 where we don't need injective resolutions, and which can be represented faithfully in a weak meta theory

The uniformity and elegance of this picture is well expressed using the language of dependent types

Summary

If we apply this to the sheaf/stack model of dependent types, we get a definition of cohomology groups of sheaves of abelian group, with rather direct proofs of the basic properties (e.g. long exact sequence of cohomology groups associated to a short exact sequence)

This can be formally represented in dependent type theory, and the formal proofs are quite close to the informal proofs

Example

Let F be a field and X be the étale site of finite separable extensions of F

This is the *small étale topos* over F

This site is quite natural as a constructive representation of the separable closure of F

See *Dynamic Newton-Puiseux Theorem*, Th. C. and B. Mannaa, 2013

We have a sheaf L on X which represents the *separable closure* of F, and the sheaf G_m which represents the invertible elements of L

Example

We can prove constructively that $H^1(X, G_m)$ is trivial, by showing that any global G_m-torsor is trivial

We expect to be able to prove constructively that $H^2(X, G_m)$ is the Brauer group of F, i.e. the abelian group of algebras that are locally matrix algebra, modulo Morita equivalence

We can start from a ring R and define the étale topos over R

A R-algebra which is locally a matrix algebra is called an *Azumaya algebra*

We expect to be able to prove constructively that the torsion part of $H^2(X, G_m)$ is the Brauer group of R

Milnor's conjecture

We can formulate Milnor's conjecture in a weak meta theory: the canonical map $K_n^M(F)/2 \rightarrow H^n(X, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism

Milnor's conjecture is about the Eilenberg-MacLane space of $\mathbb{Z}/2\mathbb{Z}$ in this topos