Inductive Definitions and Constructive Mathematics

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This talk

Survey some use of inductive definitions in *constructive* mathematics

Generalised inductive definitions

-ordinals

-formal/point-free topology

-σ-complete Boolean algebra

-Cantor-Bendixson as analysed by Lorenzen

-open induction and infinitary combinatorics

-some remarks on invariance

Generalised Inductive Definitions in Mathematics

Cantor introduction of ordinals iteration of F' operation, derived subset of F

Finite iteration $F^{(n)}$ and then introduces symbol $F^{(\omega)} = \cap_n F^{(n)}$

Cantor-Bendixson, a special case of the continuum hypothesis

A closed subset of $[0,1]$ is either countable of of cardinality $2^{\aleph_0}$

Borel's introduction of Borel sets, starting from open intervals (r,s) and iterating the operation of countable disjoint union, and the operation $B - A$ for $A \subseteq B$

Inductive Definitions in Logic

Frege *Begriffsschrift*, 1879

Transitive closure of a relation defined using an impredicative quantification

Generalized to transfinite induction in *Principia Mathematica*, 1912

Inductive Definitions in Type Theory

Hauptsatz for the intuitionistic theory of iterated inductive definitions, 1971

On the Strength of Intuitionistic Reasoning
talk at the Bucharest conference 1971, August 29 to September 4

First place where Martin-Löf hints at a connection of dependent type theory with computer science

Formulated with a type of all types, that was shown inconsistent by Girard, but the formulation can be taken as it is with a hierarchy of universes

Inductive Definitions in Type Theory

In the formal theory the abstract entities (natural numbers, ordinals, functions, types, and so on) become represented by certain symbol configurations, called terms, and the definitional schema, read from the left to the right, become mechanical reduction rules for these symbol configurations. Type theory effectuates the computerization of abstract intuitionistic mathematics that above all Bishop has asked for.

It provides a framework in which we can express *conceptual* mathematics in a *computational* way.

All computations are expressed in a *fixed* programming language (general recursion and case analysis)

Inductive Definitions in Type Theory

We can introduce the type N, the type of natural numbers. 0 is an object of type N and, if n is an object of type N, so is its successor $n+1$.

Inductive Definitions in Type Theory

Given an object c of type $C(0)$ and a function g of type $\Pi_{n:N} C(n) \to C(n+1)$ we may introduce a function f of type $\Pi_{n:N} C(n)$ by the recursion schema

$$f(0) = c \qquad f(n+1) = g(n, f(n))$$

Inductive Definitions in Type Theory

Thinking of $C(x)$ as a proposition f is a proof of the universal proposition $\Pi_{n:N}C(n)$ which we get by applying the principle of mathematical induction In the case $C(x)$ does not depend explicitely on x we get the schema of primitive recursion (at higher types), schema introduced by Hilbert and used later by Gödel

Inductive Definitions in Type Theory

We can introduce the type Ord, the type of ordinal numbers. 0 is an object of type Ord and, if x is an object of type Ord, so is its successor $x+1$ and if u is a function of type $N \rightarrow Ord$ then its limit $\mathrm{lim}\ u$ is an object of type Ord

Inductive Definitions in Type Theory

Given an object c of type $C(0)$ and a function g of type $\Pi_{\alpha:Ord} C(\alpha) \to C(\alpha+1)$ and h a function of type $\Pi_{u:N\to Ord}(\Pi_{n:N} C(u(n))) \to C(\mathsf{lim}\ u)$ we may introduce a function f of type $\Pi_{\alpha:Ord} C(\alpha)$ by the recursion schema

$$f(0) = c \quad f(\alpha + 1) = g(\alpha, f(\alpha)) \quad f(\mathsf{lim}\ u) = h(u, f \circ u)$$

where $(f \circ u)(n) = f(u(n))$

Inductive Definitions in Type Theory

Thinking of $C(\alpha)$ as a proposition, f is a proof of the universal proposition $\Pi_{\alpha:Ord}C(\alpha)$ which we get by applying the principle of transfinite induction over the second number class ordinals.

Iterated Inductive Definitions in Type Theory

Much of the literature on recursive ordinals is done nonconstructively. Closer inspection shows that this *theory is much more elegant if done intuitionistically.*

Kreisel in Lectures on modern mathematics, vol. 3, edited by T. L. Saaty, 1965

Example: slow growing versus fast growing hierarchy

Hardy hierarchy $h\ 0\ n = n,\ h\ (x+1)\ n = h\ x\ (n+1),\ h\ \alpha\ n = h\ \alpha_n\ n$

Introduced by Hardy 1903, to show $\aleph_1 \leqslant 2^{\aleph_0}$

$h\ \omega\ n = 2n$ and $h\ \omega^\omega$ of the order of Ackermann function

Slow growing hierarchy $s\ 0\ n = 0,\ s\ (x+1)\ n = (s\ x\ n) + 1,\ s\ \alpha\ n = s\ \alpha_n\ n$

$s\ \omega\ n = n$ and $s\ (\omega + \omega)\ n = 2n$ and $s\ \omega^\omega\ n = n^n$

Given α can we find β such that $h\ \alpha = s\ \beta$? (Wainer, Girard)

S. Wainer conjectured that this holds for $\alpha = \beta = \Gamma_0$

Example: slow growing versus fast growing hierarchy

0	0,1,2,3,4,5,6,7,8,. . .
1	1,2,3,4,5,6,7,8,9,. . .
2	2,3,4,5,6,7,8,9,10,. . .
3	3,4,5,6,7,8,9,10,11,. . .
. . .	. . .
ω	0,2,4,6,8,10,12,14,16,. . .
$\omega+1$	2,4,6,8,10,12,14,16,18,. . .
. . .	. . .

Example: slow growing versus fast growing hierarchy

0	0,0,0,0,0,0,0,0,0,. . .
1	1,1,1,1,1,1,1,1,1,. . .
2	2,2,2,2,2,2,2,2,2,. . .
3	3,3,3,3,3,3,3,3,3,. . .
4	4,4,4,4,4,4,4,4,4,. . .
. . .	. . .
ω	0,1,2,3,4,5,6,7,8,. . .
$\omega+1$	1,2,3,4,5,6,7,8,9,. . .
. . .	. . .

Iterated Inductive Definitions in Type Theory

J.-Y. Girard showed that we have to go much further, e.g. h_{ϵ_0} is $s_{\varphi_{\epsilon_{\Omega+1}}(0)}$ and we need to introduce a more complex inductively defined data type

The first point at which the slow growing hieararchy catches up with Hardy hierarchy is the ordinal of the theory $ID_{<\omega}$, of arbitrary finite iterations of an inductive definition

Formally, no problem in introducing Ord_2 with constructors 0, $\mu + 1$ for $\mu : Ord_2$, lim u for $u : N \rightarrow Ord_2$ and Lim v for $v : Ord \rightarrow Ord_2$

Iterated Inductive Definitions in Type Theory

Already in Lorenzen's work, but for defining predicates

Suggested in Tait's paper *Constructive Reasoning*

For any type X type of trees branching over X

Used in Scott's paper *Constructive Validity*, and later generalized by the W type in Martin-Löf's paper 1979

Type $T(X)$ with constructors $0 : T(X)$ and $u^{+} : T(X)$ for $u : X \rightarrow T(X)$

Example: slow growing versus fast growing hierarchy

Define $S\ \mu\ n : Ord$ for $\mu : Ord_2$ by induction

$S\ 0\ n = 0 \qquad S\ (\mu+1)\ n = (S\ \mu\ n)+1 \qquad S\ (\mathsf{lim}\ u)\ n = S\ (un)\ n$

$S\ (\mathsf{Lim}\ v)\ n = \mathsf{lim}\ (\lambda_p S\ (vp)\ n)$

Define $H\ \mu\ \alpha : Ord$ for $\mu : Ord_2$ by induction

$H\ 0\ \alpha = \alpha \qquad H\ (\mu+1)\ \alpha = H\ \mu\ (\alpha+1) \qquad H\ (\mathsf{lim}\ u)\ \alpha = \mathsf{lim}\ (\lambda_n H\ (un)\ \alpha)$

$H\ (\mathsf{Lim}\ v)\ \alpha = H\ (v\alpha)\ \alpha$

Example: slow growing versus fast growing hierarchy

Define the proposition $\psi\ n\ \mu$ for $\mu : Ord_2$ by *recursion* on μ

$\psi\ n\ 0 = \top \qquad \psi\ n\ (\mu + 1) = \psi\ n\ \mu \qquad \psi\ n\ (\mathsf{lim}\ u) = \forall_m \psi\ n\ (um)$

$\psi\ n\ (\mathsf{Lim}\ v) = \forall_\alpha\ \ \psi\ n\ (v\alpha)\ \wedge\ S\ (v\alpha)\ n = S\ v(s\ \alpha\ n)\ n$

Example: slow growing versus fast growing hierarchy

Alternatively $\psi\ n\ \mu$ can be defined as an *inductively defined predicate*

$$\frac{}{\psi\ n\ 0} \qquad \frac{\psi\ n\ \mu}{\psi\ n\ (\mu+1)} \qquad \frac{\forall_m \psi\ n\ (um)}{\psi\ n\ (\mathsf{lim}\ u)}$$

$$\frac{\forall_\alpha\ \psi\ n\ (v\alpha) \qquad \forall_\alpha\ S\ (v\alpha)\ n = S\ v(s\ \alpha\ n)\ n}{\psi\ n\ (\mathsf{Lim}\ v)}$$

Example: slow growing versus fast growing hierarchy

Following the presentation in the paper of Wainer JSL 1989, we can prove

Theorem: *If* $\psi\ n\ \mu$ *then* $s\ (H\ \mu\ \alpha)\ n = h\ (S\ \mu\ n)\ (s\ \alpha\ n)$

For instance, if $\Omega = \mathsf{Lim}\ (\lambda_\alpha \alpha)$, we have $S\ \Omega\ n = n$ and $s\ \omega\ n = n$ and $H\ \Omega\ \omega = \omega + \omega$ and we can prove $\psi\ n\ \alpha$ for all $\alpha : Ord$

Hence $h\ \omega = s\ (\omega + \omega)$

One of the first example tried in the implementation of inductive types

Inductively Defined Types, 1989, Th.C. and Ch. Paulin

Formal Topology

Example in ID_2

$H\ \mu\ \alpha$ defines an element of Ord using an element of Ord_1

Seemingly impredicative definition

Non monotonic: introducing Ord_2 we can define new elements of Ord

Formal Topology

Inductive definitions are fundamental for the development of formal or point-free representation of topological spaces

Critical analysis following the Lorenzen's terminology introduced in *Logical reflection and formalism*, JSL 1958

To connect this with use of ordinals, we can cite Tait, *Constructive Reasoning*

Brouwer applied the theory of ordinals to his non-atomistic theory of non-discrete spaces such as Baire space and the continuum. But, it is just as convenient to work directly with trees, rather than with their ordering as ordinals.

Formal Topology

Martin-Löf *Notes on Constructive Mathematics*, 1970

It is written in the setting of *recursive* mathematics but most of its results hold in constructive mathematics

Cantor space $\mathcal{C}$ seen as a set of infinite binary sequences $\alpha = \alpha_0, \alpha_1, \ldots$

Point-free analysis

An open subset of Cantor space is determined by a subset $U(\sigma)$ of basic open, that we can take to be *finite* binary sequences

We can define when a basic open σ is covered by U by the inductive definition

$$\frac{U(\sigma)}{U|\sigma} \qquad \frac{U|\sigma 0 \qquad U|\sigma 1}{U|\sigma}$$

U is a bar on the tree means that we have $U|()$ for the empty sequence $()$

The closed open subsets form a Boolean algebra of propositional logic built on formal atoms $\alpha(n) = 0$

Point-free analysis

A proof of $U|()$ is a finite object

With this analysis, Heine-Borel has a direct proof: if $U|()$ holds then already a finite number of elements of U cover the space

"Any cover of Cantor space has a finite subcover"

Point-free analysis

For *Baire* space we consider sequences of natural numbers and $U|\sigma$ becomes then a *generalized* inductive definitios

$$\frac{U(\sigma)}{U|\sigma} \qquad \frac{U|\sigma 0 \qquad U|\sigma 1 \qquad U|\sigma 2 \qquad \ldots}{U|\sigma}$$

As before, U is a bar means that we have $U|()$

σ-Complete Boolean Algebra

Novikov 1943, Lorenzen 1944 (published 1951) introduced a *new* use of generalized inductive definitions

How to build the free σ-complete Boolean algebra $B \rightarrow \tilde{B}$ on a given Boolean algebra B

There, generalized inductive definitions are used twice

-for defining the elements of $\tilde{B}$ that are now formal infinitary expressions built by iterating form infinite conjunctions and disjunctions

-for defining the order relation $X \leqslant Y$ between these formal expressions that is now defined by a sequent calculus involving an infinitary rule

σ-Complete Boolean Algebra

Using this technique, Lorenzen/Novikov could show the following fundamental result

Theorem: (Lorenzen) *The initial map $B \to \tilde{B}$ is an embedding*

Lorenzen noticed that the proofs of these embedding result are *formally* similar to the (original) proof of consistency by Gentzen using a form of ω-rule

One goal of Gentzen was to explain in a constructive way the classical notion of truth of an arithmetical statement

It cannot Tarski truth semantics for a statement such as $\exists_n \forall_m f(n) \leqslant f(m)$

σ-Complete Boolean Algebra

Martin-Löf *Notes on Constructive Mathematics*

If we start from the Boolean algebra B of closed open subsets of Cantor spaces (Boolean algebra of propositional logic) then $\tilde{B}$ is the σ-complete Boolean algebra of *Borel* subsets of Cantor space

This provides a constructive way to understand what is classically inclusion between Borel subsets

In this setting a *hyper-arithmetical* proposition is defined to be an element of $\tilde{N}_2$ where N_2 is the Boolean algebra $\{0, 1\}$

σ-Complete Boolean Algebra

Both basic open sets and Borel sets are seen, not as set of points, but as purely symbolic expressions

E.g. $\alpha(0) = 0$ is the closed open subset of sequences starting with 0

$\bigwedge_n \alpha(n) = 0$ is the closed subset with only the sequence $0, 0, 0, \ldots$

σ-Complete Boolean Algebra

The set of normal sequences is

$$\bigwedge_k \bigvee_m \bigwedge_{n>m} b_{n,k}$$

where $b_{n,k}$ is the basic open expressing that

$$1/n|\Sigma_{i<n} r_i(\alpha)| < 1/2^k$$

where $r_i(\alpha) = 2\alpha_i - 1$

Lusin Measure Problem

Martin-Löf suggested to define a hyper-arithmetical real to be a Dedekind cut seen as a family of hyper-arithmetical propositions over the rationals

Following Lusin

Leçons sur les ensembles analytiques et leurs applications, 1930

he formulated the problem of defining the measure $\mu(X)$ of a Borel set as a hyper-arithmetical real in a purely inductive way

One issue is that $\mu(\bigvee_n X_n)$ is not clearly a function of $\mu(X_n)$

Lusin Measure Problem

A crucial insight of Fr. Riesz *Sur la décomposition des opérations fonctionelles linéaires*, 1928, is that the function $b \mapsto \mu(b \cap X)$ with b basic open of Cantor set, can be defined by recursion on X. Using this, we can show rather directly

Theorem (Th.C.) *It is possible to define in a purely inductive way the measure of a Borel set as a hyper-arithmetical real*

For instance, we can define the measure of the set of normal binary sequences and show that it has measure 1

See the discussion in the paper *Lorenzen and Constructive Mathematics*

Cantor-Bendixson

In this paper, we also explain the analysis of Cantor-Bendixson that Lorenzen presents in *Logical reflection and formalism*, JSL 1958 and reformulated for Cantor space $\mathcal{C}$

Classically, if F closed subset, we want to define the perfect subset $K = \cap_\alpha F^{(\alpha)}$

Lorenzen gives a point-free analysis of this situation: given $\mathcal{C} - F$ as a predicate $U(b)$ on basic open, he defines $\mathcal{C} - K$ of the form $V(b)$ where V is obtained from U by a *generalised* inductive definition

Cantor-Bendixson

This example was historically important since Kreisel, in

Analysis of Cantor-Bendixson theorem by mean of the analytic hierarchy 1959

could show that V could not be obtained “predicatively” from U, for a suitable notion of “predicativity”

Conjecture: it would be interesting to present this result in a constructive setting, giving an example with U decidable and V not hyper-arithmetic

Open induction on Cantor space

This shows the relevance of generalized inductive definitions for constructive mathematics, stressed in the paper of Lorenzen and Myhill

Constructive definition of certain analytic sets of numbers, 1959

I want now to present another another example of a property of Cantor space which involves a generalized inductive definition, the principle of *open induction*

Open induction on Cantor space

In term of points, this is the following principle

Let $<$ be the lexicographic ordering of (finite or infinite) sequences

Note that $<$ is *not* well-founded on sequences

$$1 \; > \; 01 \; > \; 001 \; > \; 0001 \; > \; 00001 \; > \; \dots$$

Open induction on Cantor space

Let U be an open subset of Cantor space $\mathcal{C}$, if we have

$$(\forall_\beta \beta < \alpha \rightarrow \beta \in U) \rightarrow \alpha \in U$$

then $U = \mathcal{C}$

Open induction on $[0, 1]$

The same principle holds on $[0, 1]$ with the usual ordering $<$

Let U be an open subset of $[0, 1]$, if we have

$$(\forall_y y < x \to y \in U) \to x \in U$$

then $U = [0, 1]$

Point-free analysis

To express that we have open induction

$$(\forall_\beta \beta < \alpha \rightarrow \beta \in U) \rightarrow \alpha \in U \qquad (*)$$

in a point-free way we need to introduce the subtree of sequences α satisfying

$$\forall_\beta \beta < \alpha \rightarrow \beta \in U$$

This is the subtree defined by the predicate

$$M(\sigma) \ = \ \forall_{\sigma'} \sigma' < \sigma \rightarrow U|\sigma'$$

Point-free analysis

$(*)$ becomes then that U is a *bar* on the subtree M

It is then direct to show by induction

Theorem (Th. C.) *If* U *is a bar on the subtree* M *then* $U|()$

Since the subtree $M(\sigma) = \forall_{\sigma'}\sigma' < \sigma \rightarrow U|\sigma'$ refers to $U|$, the notion of "bar" on the subtree M is now a *generalised* inductive definition

Open Induction on Cantor space

The principle of open induction can be formulated in the setting of intuitionistic analysis FIM, as formulated by Kleene and Vesley, and the previous result can be translated to a proof of this principle, but using bar induction on *Baire* space

Theorem (W. Veldman) *The principle of open induction implies that* ϵ_0 *is well-founded*

Open Induction on Cantor space

Technically, $BIM \vdash OI(C) \rightarrow WF(\epsilon_0)$

Using a previous result of Troesltra, this means that open induction cannot be proved only using *fan* theorem, $BIM + FAN \nvdash OI(C)$, though the principle is about Cantor space

Open induction on $[0,1]$ is equivalent to open induction on Cantor space

Open Induction on Baire space

Similarly, one can prove the principle of open induction on Baire space, but the proof uses bar induction on a tree with *uncountable* branching

Conjecture: by analogy with what happens on Cantor space, it is likely that open induction principle for Baire space cannot be proved in the system FIM of Kleene and Vesley.

Iteration of inductive definitions

This analysis of open induction was motivated by the problem of understanding constructively Nash-Williams' "minimal bad sequence" argument, used for an element proof of a Kruskal well quasi-ordering theorem on trees

G. Stolzenberg (and later D. Isles), in particular, stressed the apparent "circularity" of this seemingly impredicative argument

Interesting that this may require ID_2

Following Kreisel, systems $ID_2, ID_3, \ldots$ have been used in proof theory for consistency proof of subsystems of analysis

Iteration of inductive definitions

Tait, *Constructive Reasoning*

I have been unable to arrive at any conception of inductive definitions which distinguishes the countable case from the more general one.

I will now try to comment shortly on this issue

Remark on Invariance

How to represent mathematically the notion of “growing” collection

Extensions by sheaf models

These extensions may introduce new functions/reals

E.g. in the sheaf models on $[0,1]$ we obtain a “new” real corresponding to the injection $[0,1] \rightarrow \mathbb{R}$

So a collection such as $\mathbb{R}$ or $N \rightarrow N_2$ is not “stable” by change of base, but may grow

Remark on Invariance

Well-founded relation should not be defined in term of *universal* quantification over functions (negative) but in term of *existence* of a well-founded tree (positive)

E.g. well quasi-ordering is best formulated in terms of bar

$\forall_f \exists_n \, U(f(n))$ vs $U|()$

Remark on Invariance

Some types are stable by sheaf extension: N, N_k

Some types grow: $N \to N$, Ord, $U|\sigma$

Non monotone behavior: $(N \to N) \to N$, Ord_2, ID_2

Technically: "coherent/geometric theories are preserved by pullback along geometric morphisms between topoi"

Conclusion

Generalized inductive definitions seem essential to represent in a constructive setting some notions and results of classical mathematics

-Classical validity of arithmetical statements

-Inclusion of Borel subsets and measure of Borel subsets

-Cantor-Bendixson

The conceptual status of extensions of ID_1 needs to be further analysed